Scientific Annals of Computer Science vol. 33 (2), 2023, pp. 93–157
doi: 10.7561/SACS.2023.2.93

Describing Weighted Safety with Weighted LTL over Product ω-valuation Monoids

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Abstract

We define the notion of k-safe infinitary series over idempotent ordered totally generalized product ω-valuation monoids that satisfy specific properties. For each element k of the underlying structure (different from the neutral elements of the additive, and the multiplicative operation) we determine two syntactic fragments of the weighted LTL with the property that the semantics of the formulas in these fragments are k-safe infinitary series. For specific idempotent ordered totally generalized product ω-valuation monoids we provide algorithms that given a weighted Büchi automaton and a weighted LTL formula in these fragments, decide whether the behavior of the automaton coincides with the semantics of the formula.

Keywords: weighted safety, valuation monoids, weighted LTL

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This postdoctoral research is cofinanced by Greece and European Union (European Social Fund) through the Operational program "Development of human resources, education and lifelong learning " in the context of the Project "REINFORCEMENT OF POSTDOCTORAL RESEARCHERS- Second Cycle " (MIS 5033021) that is implemented by State Scholarship Foundation (IKY).

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1 Introduction

Verifying the reliability of information and communication systems with the use of mathematical logic has been an important research field over the last years. A successful paradigm for achieving formal verification with the use of mathematical logic is model checking. Linear Temporal Logic (LTL for short), introduced by Pnueli in [26], has been successfully used in checking the correctness of programs with important applications in industry [4]. One of the most efficient ways to develop model-checking tools incorporating LTL is the automata-theoretic approach proposed by Wolper and Vardi in [31]. In addition, LTL is a logical formalism capable of describing properties of the linear behavior of systems like safety and liveness. The former expresses the fact that "nothing bad" can happen and is a very important part of verifying the reliability of programs [4]. Safety LTL formulas (cf. [29]) over infinite words define safety infinitary languages. An infinitary language L is a safety language if for every infinite word w the following holds: if every non-empty prefix of w can be extended to a word in L, then w belongs to L as well [3].

Designing effective algorithms capable of reasoning about quantitative aspects of correctness of systems' behavior, as well as identifying and studying quantitative properties that generalize the meaning of their boolean counterparts has been, and is an active research field [24, 27, 19, 17, 5, 6, 2, 32]. The notion of weighted safety was for the first time introduced in [32]. More precisely, given a rational value v, the authors defined the notion of v-safety for series, over finite words over an alphabet A, with rational values. A series L is v-safe, if every word w, to which L assigns a value greater or equal to v, has a prefix that cannot be extended to a word over A that is assigned through L to a value less than v. A series L is weighted safe if there exists a rational value v such that L is v-safe. In that paper the authors studied the relationship of weighted safe series and weighted automata whose behavior is defined as follows: the weight of each path is obtained by adding the rational weights of the transitions forming the path, and the value that the behavior of the automaton assigns to each word is defined to be the minimum weight of all accepting paths of the weighted automaton over this specific word. By determining the structural properties of deterministic weighted automata, the authors answered decidability questions related to the weighted safety of their behavior. Multi-valued safety and liveness linear-time properties have been introduced in [19] where automata based model-checking techniques

have been developed. Quantitative safety and liveness has also been considered recently, in [17], for infinitary series over complete lattices. In that work, the authors proved important results related to the safety-liveness decomposition of the infinitary series under consideration. They also provided results related to the monitorability of infinitary series that satisfy specific approximation conditions related to safety, and co-safety. It is proved in that work that the two definitions of safe infinitary series, the one presented in [17], and the one presented in [19], are equivalent. However the definition of liveness in the two works significantly differs, and in consequence different safety-liveness decomposition results are obtained. In [5] the authors further considered a definition of threshold safety for infinitary series over complete lattices. According to that definition, an infinitary series s is threshold safe if for every value v of the domain the infinitary language consisting of all infinite words whose image through s is greater or equal to v is a safety property. They prove that for totally ordered domains the definition of threshold safety and the definition of safety presented in [17] are equivalent. For such domains they provide a topological characterization of their safe infinitary series. They also provide decidability results for the safety of well known classes of quantitative automata.

In this paper, we firstly aim to define, given a specific threshold k of a weight domain, a notion of weighted safety with respect to the given threshold for infinitary series that is capable of applying to a variety of fields. For the domain of weights we choose idempotent ordered totally generalized product ω-valuation monoids that satisfy specific properties. Infinitary series express the quantitative behavior of infinite executions of systems, and the k-safe infinitary series that we introduce in this work can be understood as series that describe systems whose behavior is characterized by the existence of a "quantitative safe" area. This area can be described as an infinitary safety language that is formed by all infinite words whose coefficient (on the series describing the system's behavior) satisfy a specific quantitative property, more specifically exceeds the given threshold (according to the order relation induced by the idempotency of the monoid). We aim to describe this notion of safety with weighted *LTL* formulas, and for specific idempotent ordered totally generalized product ω-valuation monoids provide algorithms that given a weighted Büchi automaton and a weighted *LTL* formula whose semantics is k-safe, decide whether the behavior of the automaton coincides with the semantics of the formula.

To motivate the choice of the underlying weight structure we recall that

the theory of weighted automata and weighted logics over semirings has important practical applications [12], however the spectrum of applications of weighted models can be broadened by considering weighted models and weighted specification languages over abstract algebraic structures that are not necessarily semirings. To that direction in [7], Chatterjee, Doyen and Henzinger considered weighted automata over the real numbers where the weight of a run over a finite (infinite) word is computed with the use of a function that assigns a real value to every finite (infinite) run. For finite words, examples of such functions are the functions that compute the average value of real numbers and the function of sum of real numbers, while for infinite words they used functions as Sup, LimSup, LimInf, Limit Average and Discounted Sum. With the use of the above functions we can model a wide range of procedures of systems. As is stated in [7] the peak of energy consumption can be attributed as the maximum of a sequence of weights that represent energy consumption, energy use can be attributed through the sum of real numbers, while for average response time we use the function of Limit Average. In [7] the authors presented decidability results and results concerning the expressive power of the model for different functions. Similar results were presented in [8, 9, 10] where other kinds of automata that use functions for the computation of the weight of a run were presented. In [13, 14], Droste and Meinecke, proposed the structures of valuation monoids, and ω-valuation monoids as formalisms capable of describing how these automata work for different functions. Valuation monoids (resp. ω-valuation monoids) can be described as monoids (resp. complete monoids) equipped with a valuation function (resp. ω-valuation function) such that additionally the additive operation of the monoid and the valuation function (resp. the ω-valuation function) satisfy specific properties. In [14] the authors introduced a weighted MSO over product valuation monoids, and over product ω-valuation monoids (i.e., valuation, and ω-valuation monoids further equipped with a product operation) and under the consideration of specific algebraic properties for the weight structure proved expressive equivalences of fragments of that logic with weighted automata over finite, and infinite words. In [22] a weighted *LTL* over product ω-valuation monoids, and generalized product ω-valuation monoids was presented. In that paper, an effective translation of formulas of a syntactic fragment of the weighted *LTL* to weighted generalized Büchi automata with ε-transitions was presented following the construction of [20]. We recall that the classes of generalized product ω-valuation monoids, and of product ω-valuation monoids introduced

in [22] are subclasses of the class of product ω-valuation monoids defined in [14, Definition 5.1]. Moreover, the class of generalized product ω-valuation monoids presented in [22], properly contains the class of product ω-valuation monoids considered in the same work [22].

We now present how the content of this paper is structured. In Section 2, we compare the work presented in this paper with related literature. In Section 3, we present some preliminary notions. In Section 4, we define the class of totally generalized product ω-valuation monoids that properly contains the class of generalized product ω-valuation monoids. We conclude that idempotent totally generalized product ω-valuation monoids preserve basic properties of idempotent generalized product ω-valuation monoids. We adopt the definitions of weighted generalized Büchi automata with ε-transitions, and weighted Büchi automata with ε-transitions introduced in [22], whithin the framework of totally generalized product ω-valuation monoids, and also conclude that the result of their expressive equivalence with weighted Büchi automata holds for these weight structures as well. In Section 5, we present the weighted LTL over idempotent ordered totally generalized product ω-valuation monoids, which is defined in the same way as the weighted LTL in [22]. We identify a syntactic fragment of the weighted LTL, the fragment of $\vee$-totally restricted U-nesting weighted LTL formulas, such that each formula of this fragment can be effectively translated to a weighted Büchi automaton recognizing its semantics. For this translation we follow the construction and proof of the corresponding result presented in [22] for totally restricted U-nesting LTL formulas over idempotent ordered generalized product ω-valuation monoids. If K is an idempotent ordered totally generalized product ω-valuation monoid, and $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, we define the notion of k-safe infinitary series, and for all $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, we determine two syntactic fragments of the weighted LTL, namely the fragment of k-safe totally restricted U-nesting weighted LTL formulas, and the fragment of k-safe $\vee$-totally restricted U-nesting weighted LTL formulas, with the property that the semantics of the formulas in these fragments are k-safe infinitary series. Moreover, we conclude that for every formula in these fragments we can effectively construct a weighted Büchi automaton recognizing its semantics. In Section 6, we present an example on how weighted LTL formulas with k-safe semantics can be used to argue about aspects of the quantitative behavior of systems. In Section 7, for two specific idempotent ordered totally generalized product ω-valuation monoids (resp. a specific idempotent ordered generalized product ω-valuation monoid), we

present an algorithm that given a weighted Büchi automaton and a k-safe $\vee$-totally restricted U-nesting weighted LTL formula (resp. k-safe totally restricted U-nesting weighted LTL formula), decides if the semantics of the formula coincides with the behavior of the automaton. The core of the algorithm is based on the fact that the weighted LTL formula can be effectively translated to a weighted Büchi automaton, and the quantitative language equivalence problem is decidable for weighted Büchi automata over these specific structures. For the proof of this last decidability result we follow the approach of the constructive proof of the corresponding result in [7]. We obtain as a corollary that the results of this section can be generalized for specific families of totally generalized, and generalized product ω-valuation monoids. In Section 8, we present conclusions and future work directions.

2 Related Work

As it is mentioned in the introduction, the notion of weighted safety was introduced for finitary series over the set of rational numbers in [32]. As the authors proved, it follows by that definition that for every rational number v, a series s is v-safe iff the language consisting of all words whose coefficient on s is less than v is a safety language. In the current work, given an idempotent ordered totally generalized product ω-valuation monoid K, and a $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, we obtain that an infinitary series r is k-safe iff the language consisting of all infinite words whose coefficient on r is greater or equal to k (according to the order relation induced by the idempotency of the monoid) is an infinitary safety language. We see this as a generalization of the notion of safety introduced in [32], since a corresponding result for infinitary series, as the one that is proved in [32], could be obtained by our definition if the domain of K is $\mathbb{Q} \cup \{\infty, -\infty\}$, and the natural order $\geq_K$ induced by the idempotency of K coincides with the following order relation: for every $k_1, k_2 \in K$, $k_1 \geq_K k_2$ iff $\min(k_1, k_2) = k_1$. Moreover, if we adopt the definition of threshold safety introduced in [5] in the framework of our weight structure, then by Example 19 we conclude that if an infinitary series is k-safe for some specific k, then it is not necessarily threshold safe. However, the converse statement is true, i.e., threshold safety implies k-safety for every element k of the value domain for which k-safety is defined. It is the author's understanding that in the framework of weighted LTL the definition of k-safety allows the description of more refined quantitative properties.

A weighted LTL over infinite words over the max-plus semiring with

discounting parameters was defined in [21, 20]. In that work, the author presented an effective translation of a syntactic fragment of the formulas of that logic to equivalent weighted Büchi automata. For this translation the author followed the approach of [31], i.e., the weighted *LTL* formula was firstly effectively translated to a generalized weighted Büchi automaton with ε-transitions that simulates the inductive computation of the semantics of the formula by relating the states of the automaton with the syntax of the given formula. Then, the generalized weighted Büchi automaton with ε-transitions is effectively translated to an equivalent weighted Büchi automaton. We note that generalized weighted Büchi automata with ε-transitions, and weighted Büchi automata with ε-transitions over the max-plus semiring with discounting were introduced in [21, 20], and their expressive equivalence with weighted Büchi automata was proved. This result, and the result of the effective translation of weighted *LTL* formulas to equivalent weighted Büchi automata have been generalized for a subclass of idempotent (ordered generalized) product ω-valuation monoids in [22], and in the current work we further generalize it for a subclass of idempotent ordered totally generalized product ω-valuation monoids following the arguments of the proof of [22]. To our knowledge a quantitative *LTL* was for the first time introduced in [18] where the authors defined a weighted *LTL* and weighted automata over de Morgan Algebras and an effective translation of the formulas of that logic to weighted automata was presented. We note that the core of the definition of the weighted *LTL* introduced in [21, 20, 22] goes back to the definition of weighted MSO logics over semirings by Droste, and Gastin in [11].

In [7] the authors deal with weighted automata over infinite words that use as valuation functions the classical lim sup, lim inf, and sup of infinitary sequences of rational numbers, and prove that the language equivalence problem is decidable for these types of automata. In Section 7, we prove a corresponding result for weighted Büchi automata over K_2, K_1, K_3, where K_1, K_2, K_3 are specific totally generalized product ω-valuation monoids over the domain $\mathbb{Q} \cup \{\infty, -\infty\}$ that incorporate the functions lim inf, lim sup, sup respectively in the definition of their ω-valuation functions. As it is stated in the introduction, for the proof of this result, we follow the constructive approach of the proof of the corresponding result in [7]. Core steps of the algorithms presented in Section 7 are based on basic constructive steps of that proof.

Finally, in [30] the authors presented an algorithm that given a weighted transition system that models the motion of a robot, and an *LTL* formula

that specifies the mission of the robot, determines a path of the robot that satisfies the specification, and at the same time minimizes the value of a given cost function. Weights on the transitions of the weighted transition system represent time, and the cost function is defined so that it expresses, for every possible route of the robot, the maximum time needed between instances of its movement that satisfy a specific optimizing atomic proposition. The *LTL* formula is defined in a way, so that every path that satisfies it, satisfies the optimizing atomic proposition infinitely often. The example that we present in Section 6 is motivated by the need to argue about other quantitative aspects of the behavior of systems as the one presented in [30], and to answer questions related to correctness, and trustworthiness [28].

3 Preliminaries

Let A be an alphabet, i.e., a finite non-empty set. As usually, we denote by A^* the set of all finite words over A and we let $A^+ = A^* \setminus \{\varepsilon\}$, where ε is the empty word. The set of all infinite sequences with elements in A, i.e., the set of all infinite words over A, is denoted by A^ω. Let $w \in A^\omega$. A word $v \in A^\omega$ is called a suffix of w, if $w = uv$ for some $u \in A^*$. Similarly, a word $u \in A^*$ is a called a prefix of w if $w = uv$ for some $v \in A^\omega$. Every infinite word $w = a_0 a_1 \ldots$ with $a_i \in A\,(i \geq 0)$ is written also as $w = w\,(0)\,w\,(1)\ldots$ where $w\,(i) = a_i\,(i \geq 0)$. The word $w_{\geq i}$ denotes the suffix of w that starts at position i, i.e., $w_{\geq i} = w\,(i)\,w\,(i+1)\ldots$, while the word $w_{<i}$ denotes the prefix of w, $w_{<i} = w\,(0)\ldots w\,(i-1)$. Clearly, $w_{<0} = \varepsilon$.

Let C, K be sets. We denote by $B \subseteq_{fin} C$ the fact that B is a finite subset of C and we let $(C_{fin})^\omega = \bigcup_{B \subseteq_{fin} C} B^\omega$. We denote by $\mathcal{P}(C)$ the powerset of C. An index set I of C is a subset of C. A family of elements of K over the index set I, denoted by $(k_i)_{i \in I}$, is a mapping f from I to K where $k_i = f(i)$ for all $i \in I$. We shall denote by $\mathbb{N}$ the set of non-negative integers.

We recall now the definition of Büchi, Muller, and Rabin automata over A. A Büchi automaton (Ba for short) over the alphabet A is a tuple $\mathcal{A} = (Q, A, I, \Delta, F)$ where Q is a finite state set, $\Delta \subseteq Q \times A \times Q$ is the set of transitions, $I \subseteq Q$ is the set of initial states, and $F \subseteq Q$ is the set of final states. Let $w = a_0 a_1 \ldots \in A^\omega$ with $a_i \in A\,(i \geq 0)$. A path P_w of $\mathcal{A}$ over w is an infinite sequence of transitions $P_w = (q_j, a_j, q_{j+1})_{j \geq 0}$. We let $In^Q(P_w)$ be the set of states that appear infinitely often along P_w. We will call P_w success-

ful if $q_0 \in I$, and at least one final state appears infinitely often along P_w, i.e., if $q_0 \in I$, and $In^Q(P_w) \cap F \neq \emptyset$. The behavior of $\mathcal{A}$ is defined to be the infinitary language $\|\mathcal{A}\| = \{w \in A^\omega \mid \text{there exists a successful path of } \mathcal{A} \text{ over } w\}$. A (nondeterministic) Muller automaton (Ma for short) over the alphabet A is a tuple $\mathcal{A} = (Q, A, p_0, \Delta, \mathcal{F})$, where Q is a finite set of states, $p_0 \in Q$ is the initial state, $\Delta \subseteq Q \times A \times Q$ is the set of transitions, and $\mathcal{F} \subseteq \mathcal{P}(Q)$ is the set of final subsets. A path P_w of $\mathcal{A}$ over w, and the set $In^Q(P_w)$ are defined as in the case of Büchi automata. We say that the path P_w is successful, if $q_0 = p_0$ and $In^Q(P_w) \in \mathcal{F}$. The behavior of $\mathcal{A}$ is again defined to be the infinitary language $\|\mathcal{A}\| = \{w \in A^\omega \mid \text{there exists a successful path of } \mathcal{A} \text{ over } w\}$. Finally, a (nondeterministic) Rabin automaton (Ra for short) is a tuple $\mathcal{A} = (Q, A, p_0, \Delta, \mathcal{R})$, where Q is a finite set of states, $p_0 \in Q$ is the initial state, $\Delta \subseteq Q \times A \times Q$ is the set of transitions, and $\mathcal{R} = (L_j, U_j)_{j \in J}$ is a finite family of pairs of sets of states. Again, a path P_w of $\mathcal{A}$ over w, and the set $In^Q(P_w)$ are defined as in the case of Büchi automata. We will say that the path P_w is successful if $q_0 = p_0$, and there exists an index $j \in J$ such that $In^Q(P_w) \cap L_j = \emptyset$, and $In^Q(P_w) \cap U_j \neq \emptyset$. The behavior of $\mathcal{A}$ is defined as in the previous two cases of Büchi, and Muller automata. We recall the following theorem that states the expressive equivalence of Muller, Büchi and Rabin automata.

Theorem 1 *(i) For every Muller automaton over A we can construct a Büchi automaton over A with the same behavior, and vice-versa.*

(ii) For every Rabin automaton over A we can construct a Büchi automaton over A with the same behavior, and vice-versa.

For a detailed reference on the theory of Büchi, Muller, and Rabin automata we refer to [25].

Next, we recall the definition of classical *LTL*, and the definition of safety properties (also equivalently called safety languages) and safety formulas (see [29]).

Definition 2 ([3]) *Let A be an alphabet and $L \subseteq A^\omega$. L is a safety property iff for every $w \in A^\omega$ the following hold: If $\forall i > 0, \exists u \in A^\omega$ such that $w_{<i}u \in L$, then $w \in L$.*

Example 3 *Let $A = \{a, b\}$ be an alphabet, $L = a^+b^+A^\omega \subseteq A^\omega$, and $F = a^+A^\omega \subseteq A^\omega$. We consider the infinite word $w = a^\omega$. Then, for every $i > 0$,*

$w_{<i}ba^\omega \in L$, however $w \notin L$. This implies that L is not a safety property. We can easily conclude that F is a safety property. More precisely, let $\overline{w} \in A^\omega$ such that for every $i > 0, \exists u \in A^\omega$ such that $\overline{w}_{<i}u \in F$. Then, $w(0) = a$, which implies that $\exists j > 0, \overline{u} \in A^\omega$ such that $\overline{w} = a^j\overline{u}$, i.e., $\overline{w} \in F$.

Let AP be a finite set of atomic propositions. We shall denote the elements of AP by $a, b, c, \ldots$. The syntax of classical LTL over AP is given by the grammar

$$\varphi ::= true \mid a \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \bigcirc\varphi \mid \varphi U \varphi \mid \Box\varphi$$

where $a \in AP$. We shall denote by $LTL(AP)$ the class of all LTL formulas over AP. As usual we identify $\neg\neg\varphi$ with φ for every $\varphi \in LTL(AP)$. Let now $\varphi \in LTL(AP)$, and $w \in (\mathcal{P}(AP))^\omega$. Then, for every $i \geq 0$, it is defined what it means that φ holds in w at position i, which is denoted by $w, i \models \varphi$, inductively in the following way:

- $w, i \models true$
- $w, i \models a$, if $a \in w(i)$
- $w, i \models \neg\varphi$, if $w, i \not\models \varphi$
- $w, i \models \bigcirc\varphi$, if $w, i+1 \models \varphi$
- $w, i \models \varphi U \psi$, if there exists $j \geq i$ such that $w, i' \models \varphi$ for all $i \leq i' < j$, and $w, j \models \psi$.
- $w, i \models \varphi \vee \psi$, if $w, i \models \varphi$ or $w, i \models \psi$
- $w, i \models \varphi \wedge \psi$, if $w, i \models \varphi$ and $w, i \models \psi$
- $w, i \models \Box\varphi$, if for every $j \geq i$, $w, j \models \varphi$

For an infinite word $w \in (\mathcal{P}(AP))^\omega$, we will say that φ holds in w, and we will denote by $w \models \varphi$, iff $w, 0 \models \varphi$. We say that two LTL formulas φ, ψ are equivalent and denote by $\varphi \equiv \psi$ iff for every $w \in (\mathcal{P}(AP))^\omega$, $w \models \varphi$ iff $w \models \psi$. We also recall that the *weak until* operator $\widetilde{U}$ is defined in the following way $\varphi \widetilde{U} \psi := \Box\varphi \vee (\varphi U \psi)$. For every $\varphi \in LTL(AP)$ we let the *language of* φ, denoted by $\mathcal{L}(\varphi)$, be defined as follows: $\mathcal{L}(\varphi) = \{w \in (\mathcal{P}(AP))^\omega \mid w \models \varphi\}$.

A formula φ is a safety formula iff $\mathcal{L}(\varphi)$ is a safety property over $\mathcal{P}(AP)$. We let $sLTL(AP)$ be the fragment of $LTL(AP)$ given by the following grammar

$$\varphi ::= true \mid a \mid \neg a \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \bigcirc\varphi \mid \varphi \widetilde{U} \varphi \mid \Box\varphi$$

where $a \in AP$.

Theorem 4 ([29]) *For every $a \in AP, a, \neg a$ are safety formulas and if φ, ψ are safety formulas, then so are $\varphi \wedge \psi, \varphi \vee \psi, \bigcirc\varphi, \varphi \widetilde{U} \psi$, and $\Box\varphi$.*

It follows by the previous lemma that every formula in $sLTL(AP)$ is a safety formula.

We turn now to monoids and infinitary series. A monoid $(K, +, \mathbf{0})$ is an algebraic structure equipped with a non-empty set K and an associative additive operation $+$ with a zero element $\mathbf{0}$, i.e., $\mathbf{0} + k = k + \mathbf{0} = k$ for every $k \in K$. The monoid K is called commutative if $+$ is commutative.

A monoid $(K, +, \mathbf{0})$ is called complete if it is equipped, for every index set I, with an infinitary sum operation $\sum_I : K^I \to K$ such that for every family $(k_i)_{i \in I}$ of elements of K we have

$$\sum_{i \in \emptyset} k_i = \mathbf{0}, \sum_{i \in \{j\}} k_i = k_j, \sum_{i \in \{j,l\}} k_i = k_j + k_l \text{ for } j \neq l$$

and

$$\sum_{j \in J} \left(\sum_{i \in I_j} k_i \right) = \sum_{i \in I} k_i,$$

if $\bigcup_{j \in J} I_j = I$ and $I_j \bigcap I_{j'} = \emptyset$ for $j \neq j'$. We note that every complete monoid is commutative.

Let K be a complete monoid. K is called additively idempotent (or simply idempotent), if $k + k = k$ for every $k \in K$. Furthermore, K is zero-sum free if $k + k' = \mathbf{0}$ implies $k = k' = \mathbf{0}$. It is well known that if K is idempotent, then K is necessarily zero-sum free [1]. We recall (cf. [16]) that idempotency gives rise to a natural partial order in K defined in the following way. Let $k, k' \in K$, then $k \leq k'$ iff $k' = k' + k$. Equivalently, it holds $k \leq k'$ iff $k' = k'' + k$ for some $k'' \in K$ (cf. [12, Chapter 5]). We shall denote by $k < k'$ the fact that $k \leq k'$, and $k \neq k'$. We shall call the natural order of K a total order, if $k \leq k'$, or $k' \leq k$ for all $k, k' \in K$. In this case we will call K an idempotent ordered complete monoid. Clearly, if K is an idempotent ordered complete monoid it holds: For every $k, k' \in K$, if $k \leq k'$ is not true, then $k' < k$. Let A be an alphabet and K be an idempotent ordered complete monoid. An infinitary series over A and K is a mapping $s : A^\omega \to K$. For every $w \in A^\omega$ we write (s, w) for the value $s(w)$ and refer to it as the coefficient of s on w. We denote by $K\langle\langle A^\omega \rangle\rangle$ the class of all infinitary series over A and K. For a series $s \in K\langle\langle A^\omega \rangle\rangle$ we let $Im(s) = \{(s, w) \mid w \in A^\omega\}$, $supp(s) = \{w \in A^\omega \mid (s, w) \neq \mathbf{0}\}$, and

$L_{\geq k}(s) = \{w \in A^\omega \mid (s,w) \geq k\}$ for every $k \in K$. Moreover, for every $k \in K$, we shall denote by k, the infinitary series over A and K defined by $(k,w) = k$ for every $w \in A^\omega$. Given two series $s_1, s_2 \in K\langle\langle A^\omega \rangle\rangle$ we write $s_1 \sqsubseteq s_2$ if $(s_1,w) \leq (s_2,w)$ for all $w \in A^\omega$, and $s_1 = s_2$ if $(s_1,w) = (s_2,w)$ for all $w \in A^\omega$. The quantitative inclusion problem, for the series s_1, s_2, is to decide whether $s_1 \sqsubseteq s_2$, while the quantitative language-equivalence problem decides whether $s_1 = s_2$.

Next, we recall from [30] the definition of weighted transition systems. $\mathbb{R}_{>0}$ stands for the set of positive real numbers.

Definition 5 *A weighted transition system is a tuple* $T = (Q, Q_0, R, AP, L, we)$, *where* Q *is finite set of states,* $Q_0 \subseteq Q$ *is the set of initial states,* $R \subseteq Q \times Q$ *is the transition relation,* AP *is a finite set of atomic propositions,* $L : Q \to \mathcal{P}(AP)$ *is a labeling function, and* $we : R \to \mathbb{R}_{>0}$ *is a weight function.*

A run r of T is an infinite sequence of states $r = q_0q_1q_2\ldots$ with $q_0 \in Q_0$, and $(q_i, q_{i+1}) \in R$ for every $i \geq 0$. Every run $r = q_0q_1\ldots$ defines a unique word $w_r = L(q_0)L(q_1)L(q_2)\ldots \in (\mathcal{P}(AP))^\omega$, and a unique sequence of weights $\omega_r = (we(q_i, q_{i+1}))_{i\geq 0}$.

In the rest of the paper A will stand for an alphabet, and AP for a finite set of atomic propositions.

4 Totally Generalized Product ω-valuation Monoids

We aim to identify structures equipped with ω-valuation functions that are well studied and can be used for practical applications. To this end, we consider a larger subclass of product ω-valuation monoids, than this of generalized product ω-valuation monoids (for short GP-ω-valuation monoids) introduced in [22, Definition 2]. More precisely, we define totally generalized product ω-valuation monoids. These are defined as GP-ω-valuation monoids with the only difference that we restrict the requirement of the validity of the distributivity of the ω-valuation function over finite sums to a smaller range of families of elements of the value domain. This allows the identification of examples of totally generalized product ω-valuation monoids that incorporate classical ω-valuation functions like *sup* and limsup of countably infinite families of rational numbers. In [22], the author considered GP-ω-valuation monoids that satisfy additional properties, sufficient to establish the effective

translation of formulas of fragments of the weighted *LTL*, with weights over GP-ω-valuation monoids, to weighted Büchi automata. In this work, we will identify sufficient properties that totally generalized product ω-valuation monoids should satisfy, so that we can determine fragments of the weighted *LTL* whose formulas' can be effectively translated to equivalent weighted Büchi automata and their semantics are k-safe infinitary series ($k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$). For the translation of our weighted *LTL*-formulas to equivalent weighted Büchi automata the properties considered in [22] are sufficient in the case of totally generalized product ω-valuation monoids as well. We further equip the structure with properties related to the total ordering of the monoid and the monotononicity of the ω-valuation function, so that we can identify fragments of the weighted *LTL* whose formulas' semantics are k-safe infinitary series. Whether the properties considered to obtain the aforementioned results are necessary it remains open.

Definition 6 *An ω-valuation monoid $(K, +, Val^\omega, \mathbf{0})$ is a complete monoid $(K, +, \mathbf{0})$ equipped with an ω-valuation function $Val^\omega : (K_{fin})^\omega \to K$ such that $Val^\omega (k_i)_{i\in\mathbb{N}} = \mathbf{0}$ whenever $k_i = \mathbf{0}$ for some $i \geq 0$. A totally generalized product ω-valuation monoid (for short TGP-ω-valuation monoid) $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$ is an ω-valuation monoid $(K, +, Val^\omega, \mathbf{0})$ further equipped with a product operation $\cdot : K^2 \to K$, with $\mathbf{1} \in K$ such that $Val^\omega (\mathbf{1}^\omega) = \mathbf{1}$ and $\mathbf{0} \cdot k = k \cdot \mathbf{0} = \mathbf{0}$, $\mathbf{1} \cdot k = k \cdot \mathbf{1} = k$ for all $k \in K$. Additionally, for every index set I and $k \in K$, $\sum\limits_I (k \cdot \mathbf{1}) = k \cdot \sum\limits_I \mathbf{1}$, and for every $L \subseteq_{fin} K$, finite index sets $I_j\, (j \geq 0)$ such that for all $j \geq 0$, it holds $k_{i_j} \in L \setminus \{\mathbf{0}, \mathbf{1}\}$ for all $i_j \in I_j$, or $k_{i_j} \in \{\mathbf{0}, \mathbf{1}\}$ for all $i_j \in I_j$, we have*

$$Val^\omega \left(\sum_{i_j \in I_j} k_{i_j} \right)_{j\in\mathbb{N}} = \sum_{(i_j)_j \in I_0 \times I_1 \times \ldots} Val^\omega \left(k_{i_j}\right)_{j\in\mathbb{N}} \tag{1}$$

Equation 1 (in the sequel called Property 1) states the distributivity of Val^ω over finite sums that are taken over finite families of elements of K that satisfy the following property: For each family, either every element of the family belongs to $\{\mathbf{0}, \mathbf{1}\}$, or every element of the family belongs to $L \setminus \{\mathbf{0}, \mathbf{1}\}$ (where L is a finite subset of K). We recall that Property 1 has also been considered in [23] (for a larger range of families of elements of K) for the definition of Cauchy ω-indexed valuation monoids. Every GP-ω-valuation monoid is also a TGP-ω-valuation monoid. A GP-ω-valuation monoid (resp.

TGP-ω-valuation monoid) $(K, +, \cdot, Val^{\omega}, \mathbf{0}, \mathbf{1})$ is called idempotent (resp. ordered) if $(K, +, \mathbf{0})$ is an idempotent monoid (resp. an ordered monoid). The results in [22, Lemmas 1 and 2] can be proved for idempotent TGP-ω-valuation monoids by adopting the arguments of the corresponding proofs. In fact, in [22, Lemma 1] we prove properties of the infinitary sum operation $\sum$ of an idempotent GP-ω-valuation monoid K. More specifically, we prove complete idempotence under the condition that the cardinality of the index set, over which $\sum$ is applied, is at most continuum. We further prove that when $\sum$ is applied over all elements of K, and over all elements of any subset K' of K, resulting sums $\sum\limits_{k\in K} k$, $\sum\limits_{k\in K'} k$ preserve the order indicated by the subset relation, i.e., $\sum\limits_{k\in K'} k \leq \sum\limits_{k\in K} k$. Lastly, we prove that $\sum$ is compatible with the addition of subsets of K.[2] In [22, Lemma 2], (resp. [22, Lemma 3ii]) we identify conditions the elements of two subsets K', K'' of K (resp. two countably infinite families $\left(k_i^1\right)_{i\geq 0}$, $\left(k_i^2\right)_{i\geq 0}$ of elements of K) should satisfy, so that we can conclude the order relation between $\sum\limits_{k\in K'} k$ and $\sum\limits_{k\in K''} k$ (resp. between $Val^{\omega}\left(k_i^1\right)_{i\geq 0}$ and $Val^{\omega}\left(k_i^2\right)_{i\geq 0}$). Following the arguments presented in the proof of [22, Lemma 3ii] we obtain a corresponding result for idempotent TGP-ω-valuation monoids presented in the following lemma.

Lemma 7 *Let $(K, +, \cdot, Val^{\omega}, \mathbf{0}, \mathbf{1})$ be an idempotent TGP-ω-valuation monoid, and $L \subseteq_{fin} K$. If $\left(k_i^1\right)_{i\geq 0}$ and $\left(k_i^2\right)_{i\geq 0}$ are families of elements of L such that for every $i \geq 0, k_i^1 \leq k_i^2$, and for all $i \geq 0$, it holds $\left\{k_i^1, k_i^2\right\} \subseteq L \setminus \{\mathbf{0}, \mathbf{1}\}$, or $\left\{k_i^1, k_i^2\right\} \subseteq \{\mathbf{0}, \mathbf{1}\}$, then $Val^{\omega}\left(k_i^1\right)_{i\geq 0} \leq Val^{\omega}\left(k_i^2\right)_{i\geq 0}$.*

Lemma 8 *Let $(K, +, \cdot, Val^{\omega}, \mathbf{0}, \mathbf{1})$ be an idempotent ordered TGP-ω-valuation monoid, and $k, k_1, k_2 \in K$. If $k_1 < k$, and $k_2 < k$, then $k_1 + k_2 < k$.*

Proof: It holds $k_1 < k$, and $k_2 < k$. Thus, $k_1 \leq k$, and $k_2 \leq k$, Then, $k_1 \leq k$, implies $k_1 + k = k$ and since $k_2 \leq k$, we get $k_1 + (k_2 + k) = (k_1 + k_2) + k = k$, i.e., $k_1 + k_2 \leq k$. Assume now that $k_1 + k_2 = k$. Since K is ordered, it holds $k_1 \leq k_2$, or $k_2 \leq k_1$. If the first case is true, then by definition of the natural order induced by idempotency we get $k_2 = k_1 + k_2$, and thus $k_2 = k$, which is a contradiction. If the second case is true, i.e., if $k_1 \leq k$, then following the arguments of the previous case we conclude

[2] For any two subsets K',K'' of K we let the sum of K' with K'' be the set $K' + K'' = \{k' + k'' \mid k' \in K', k'' \in K''\}$.

$k_1 = k$ which is a contradiction. Thus, $k_1 + k_2 \leq k$, and $k_1 + k_2 \neq k$, i.e., $k_1 + k_2 < k$, as desired. □

In the rest of this paper we will consider idempotent ordered TGP-ω-valuation monoids $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$ that further satisfy the following properties. For all $k, k_i \in K$ $(i \geq 1)$ it holds

$$Val^\omega(\mathbf{1}, k_1, k_2, k_3, \ldots) = Val^\omega(k_i)_{i\geq 1}, \tag{2}$$

$$k = Val^\omega(k, \mathbf{1}, \mathbf{1}, \mathbf{1}, \ldots), \tag{3}$$

$$k \leq \mathbf{1}. \tag{4}$$

Moreover, for all $k \in K$, and all $(k_i)_{i\geq 0} \in (K_{fin})^\omega$ with $k_i \geq k$ $(i \geq 0)$ it holds

$$Val^\omega(k_0, k_1, k_2, \ldots) \geq k. \tag{5}$$

Properties 2, 3 have also been considered in [22]. As it was stated in [22] they express a notion of neutrality of $\mathbf{1}$ with respect to the ω-valuation function, and are sufficient to establish the effective translation of formulas of fragments of the weighted *LTL* over GP-ω-valuation monoids introduced in that work to weighted Büchi automata. Property 4 states the fact that $\mathbf{1}$ is the maximal element of the total order, while Property 5 expresses a restricted notion of monotonicity.

Example 9 *[22, Example 2] Let* $K_1 = \left(\overline{\mathbb{Q}}, \sup, \inf, \mathit{liminf}, -\infty, \infty\right)$ *where* $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty, -\infty\}$ *and liminf is an* ω*-valuation function from* $\left(\overline{\mathbb{Q}}_{fin}\right)^\omega$ *to* $\overline{\mathbb{Q}}$ *defined by*

$$\mathit{liminf}\left((d_i)_{i\geq 0}\right) = \begin{cases} -\infty & \text{if there exists } i \geq 0 \\ & \text{with } d_i = -\infty \\ \infty & \text{if for all } i \geq 0, \\ & d_i = \infty \\ \sup\limits_{i\geq 0}\left(\inf\{d_k \mid k \geq i, d_k \neq \infty\}\right) & \text{if } d_j \neq -\infty \text{ for all} \\ & j \geq 0, \text{and there exist} \\ & \text{infinitely many } i \geq 0 \\ & \text{with } d_i \neq \infty \\ \inf\{d_i \mid i \geq 0 \text{ with } d_i \neq \infty\} & \text{otherwise} \end{cases}$$

K_1 is an idempotent ordered GP-ω-valuation monoid that satisfies Properties 2, 3, 4, and 5.

Example 10 *We consider the structure* $K_2 = \left(\overline{\mathbb{Q}}, \sup, \inf, \textit{limsup}, -\infty, \infty\right)$ *where* $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty, -\infty\}$ *and limsup is an* ω*-valuation function from* $\left(\overline{\mathbb{Q}}_{fin}\right)^{\omega}$ *to* $\overline{\mathbb{Q}}$ *defined by*

$$\textit{limsup}\left((d_i)_{i\geq 0}\right) = \begin{cases} -\infty & \textit{if there exists } i \geq 0 \\ & \textit{with } d_i = -\infty \\ \infty & \textit{if for all } i \geq 0, \\ & d_i = \infty \\ \inf\limits_{i\geq 0}\left(\sup\left\{d_k \mid k \geq i, d_k \neq \infty\right\}\right) & \textit{if } d_j \neq -\infty \textit{ for all} \\ & j \geq 0, \textit{and there exist} \\ & \textit{infinitely many } i \geq 0 \\ & \textit{with } d_i \neq \infty \\ \sup\left\{d_i \mid i \geq 0 \textit{ with } d_i \neq \infty\right\} & \textit{otherwise} \end{cases}$$

K_2 is an idempotent ordered TGP-ω-valuation monoid that satisfies Properties 2, 3, 4, and 5.

Example 11 *We consider the structure* $K_3 = \left(\overline{\mathbb{Q}}, \sup, \inf, \sup_{-\infty}, -\infty, \infty\right)$ *where* $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty, -\infty\}$ *and* $\sup_{-\infty}$ *is an* ω*-valuation function from* $\left(\overline{\mathbb{Q}}_{fin}\right)^{\omega}$ *to* $\overline{\mathbb{Q}}$ *defined by*

$$\sup_{-\infty}\left((d_i)_{i\geq 0}\right) = \begin{cases} -\infty & \textit{if } \exists i \geq 0 \textit{ with } d_i = -\infty \\ \infty & \textit{if } d_i = \infty \textit{ for all } i \geq 0 \\ \sup\left\{d_k \mid k \geq 0, d_k \neq \infty\right\} & \textit{if } d_i \neq -\infty \textit{ for all } i \geq 0, \\ & \textit{and } \exists i \geq 0 \textit{ with } d_i \neq \infty. \end{cases}$$

K_3 is an idempotent ordered TGP-ω-valuation monoid that satisfies Properties 2, 3, 4, and 5.

The proofs that K_2, K_3 are idempotent ordered TGP-ω-valuation monoids that satisfy Properties 2, 3, 4, and 5 can be found in the Appendix. We also note that ω-valuation functions over the reals that incorporate classical *sup*, limsup in their definitions have also appeared in [23, 13, 14]. As it is stated in [7], for $(d_i)_{i\geq 0} \in \left(\overline{\mathbb{Q}}_{fin}\right)^{\omega}$ the value $\inf\limits_{i\geq 0}\left(\sup\left\{d_k \mid k \geq i\right\}\right) = \lim\limits_{i\geq 0}\left(\sup\left\{d_k \mid k \geq i\right\}\right)$ corresponds to the greatest

element of $\overline{\mathbb{Q}}$ that appears infinitely often along the sequence $(d_i)_{i\geq 0}$, while $\sup\Big((d_i)_{i\geq 0}\Big)$ equals the greatest element of $\overline{\mathbb{Q}}$ that appears in $(d_i)_{i\geq 0}$. Finally, the value $\sup\limits_{i\geq 0}(\inf\{d_k \mid k\geq i\}) = \lim\limits_{i\geq 0}(\inf\{d_k \mid k\geq i\})$ corresponds to the least element of $\overline{\mathbb{Q}}$ that appears infinitely often along the sequence $(d_i)_{i\geq 0}$.

Remark 12 *Let $(K,\leq)$ be a totally ordered set with a minimum, and a maximum element which are respectively denoted by $\mathbf{0}$, and $\mathbf{1}$. Then, $\big(K,\sup_{\leq},\mathbf{0}\big)$, $(K,\inf_{\leq},\mathbf{1})$ are idempotent complete monoids. $\sup_{\leq}$, $\inf_{\leq}$ stand for the supremun, and infimum respectively of possibly infinite families of elements of K, and are defined in the classical way with respect to $\leq$-ordering of K. We define the ω-valuation functions $liminf_{\leq}$, $limsup_{\leq}$, $\sup_0$ from $(K_{fin})^{\omega}$ to K in a corresponding way with the way the ω-valuation functions liminf, limsup, and $sup_{-\infty}$ are defined in Examples 9, 10, and 11 respectively. More precisely, the definitions of $liminf_{\leq}$, $limsup_{\leq}$, $\sup_0$ are obtained by the definitions of liminf, limsup, and $sup_{-\infty}$ by substituting* inf, sup, -∞, ∞ *by* $\inf_{\leq}$, $\sup_{\leq}$, $\mathbf{0}$, $\mathbf{1}$ *respectively. We consider the structures $M_1 = \big(K,\sup_{\leq},\inf_{\leq},liminf_{\leq},\mathbf{0},\mathbf{1}\big)$, $M_2 = \big(K,\sup_{\leq},\inf_{\leq},limsup_{\leq},\mathbf{0},\mathbf{1}\big)$, $M_3 = \big(K,\sup_{\leq},\inf_{\leq},\sup_{\mathbf{0}},\mathbf{0},\mathbf{1}\big)$. Then, by adopting the arguments of the proof of [22, Example 2], and of Examples 10, 11 presented in the Appendix we get that M_1 (resp. M_2, M_3) is an (resp. are) idempotent ordered GP-ω-valuation monoid (resp. idempotent ordered TGP-ω-valuation monoids) that satisfies (resp. satisfy) Properties 2, 3, 4, and 5.*

In the following example we present totally ordered sets with maximum, and minimum element.

Example 13 *Let $N = \mathbb{Z}_{>}\cup\{0,+\infty\}$, and $K = N\times N$. We consider the pair $(K,\preceq)$ where $\preceq$ stands for the order relation defined in the following way: For every $(a_1,a_2),(b_1,b_2)\in K$ we let $(a_1,a_2)\preceq(b_1,b_2)$ if $a_1<b_1$, or if $a_1=b_1$ and $a_2\leq b_2$. Clearly, $(K,\preceq)$ is totally ordered with minimum element $(0,0)$, and maximum element $(+\infty,+\infty)$.*

We consider now the pair $(K,\leqq)$ where $\leqq$ stands for the order relation defined in the following way: For every $(a_1,a_2),(b_1,b_2)\in K$ we let $(a_1,a_2)\leqq(b_1,b_2)$ if $a_1+a_2<b_1+b_2$, or if $a_1+a_2=b_1+b_2$ and $a_1\leq b_1$. Clearly, $(K,\leqq)$ is totally ordered with minimum element $(0,0)$, and maximum element $(+\infty,+\infty)$.

In the previous example elements of K could represent two distinct indications of energy consumption of a system. If $(d_i)_{i\geq 0}$ is a sequence of

measurements of these indications at distinct time moments, $\sup_{\leqq}\left((d_i)_{i\geq 0}\right)$ corresponds to the least pair (a_1, a_2) (according to $\leqq$-ordering) with the property that at every time moment the overall energy consumption is bounded by the sum $a_1 + a_2$, and whenever it reaches the value $a_1 + a_2$ the first indication of energy consumption is bounded by a_1. Alternatively, elements of K could represent two independent function indications of a device. In this case, if $(d_i)_{i\geq 0}$ is a sequence of measurements of these indications at distinct time moments, then $\sup_{\preceq}\left((d_i)_{i\geq 0}\right)$ corresponds to the least pair (b_1, b_2) (according to $\preceq$-ordering) with the property that at every time moment the first indication is bounded by b_1, and whenever it reaches b_1, the second indication is bounded by b_2. Clearly, $\preceq$-ordering, and $\leqq$-ordering of K can be used to argue about quantitative properties of different aspects of the behavior of a system whenever priority is of interest.

In the rest of the paper, we will refer to idempotent ordered GP-ω-valuation monoids (resp. idempotent ordered TGP-ω-valuation monoids) that satisfy Properties 2, 3, 4, and 5 as idempotent ordered GP-ω-valuation monoids (resp. idempotent ordered TGP-ω-valuation monoids).

In [22] weighted generalized Büchi automata with ε-transitions (ε-wgBa for short), and weighted Büchi automata with ε-transitions (ε-wBa for short) over idempotent ordered GP-ω-valuation monoids were introduced, and their expressive equivalence with weighted Büchi automata (wBa for short) over idempotent ordered GP-ω-valuation monoids was proved (see [22, Lemmas 4 and 5ii]). The definitions of ε-wgBa, ε-wBa, and wBa over GP-ω-valuation monoids in [22, Definition 3] can be generalized for idempotent ordered TGP-ω-valuation monoids in a straightforward way. The expressive equivalence of ε-wgBa, ε-wBa, and wBa over idempotent ordered TGP-ω-valuation monoids can be proved with the same arguments that are used in the proofs of [22, Lemmas 4 and 5ii]. We recall the definition of wBa. Let K be an idempotent ordered TGP-ω-valuation monoid.

Definition 14 *A weighted Büchi automaton over A and K is a quadruple $\mathcal{A} = (Q, wt, I, F)$, where Q is the finite set of states, $wt : Q \times A \times Q \to K$ is a mapping assigning weights to the transitions of the automaton, $I \subseteq Q$ is the set of initial states, and $F \subseteq Q$ is the set of final states.*

Let $w = a_0 a_1 \ldots \in A^\omega$ with $a_i \in A\,(i \geq 0)$. A path P_w of $\mathcal{A}$ over w is an infinite sequence of transitions $P_w = (q_j, a_j, q_{j+1})_{j\geq 0}$. We let the weight

of P_w be the value

$$weight_{\mathcal{A}}(P_w) = Val^{\omega}(wt(q_j, a_j, q_j))_{j\geq 0}$$

The set $In^Q(P_w)$ is defined as for Ba. The path P_w is called *successful* if $q_0 \in I$, and $In^Q(P_w) \cap F \neq \emptyset$. We shall denote by $succ_{\mathcal{A}}(w)$ the set of all successful paths of $\mathcal{A}$ over w. The behavior of $\mathcal{A}$ is the infinitary series $\|\mathcal{A}\| : A^{\omega} \to K$ with coefficients specified, for every $w \in A^{\omega}$,

$$(\|\mathcal{A}\|, w) = \sum_{P_w \in succ_{\mathcal{A}}(w)} weight_{\mathcal{A}}(P_w).$$

Example 15 *A wBa automaton $\mathcal{M}$ over K_2 and $A = \{a, b\}$ is presented in Figure 1. The behavior of $\mathcal{M}$ is defined as follows. $(\|\mathcal{M}\|, w) = \infty$ if $w = b^{\omega}$, $(\|\mathcal{M}\|, w) = 2$ if $w = a\overline{w}$ for some $\overline{w} \in A^{\omega}$ with a finite number of appearances of b, $(\|\mathcal{M}\|, w) = 3$ if $w = a\overline{w}$ for some $\overline{w} \in A^{\omega}$ with an infinite number of appearances of b, and $(\|\mathcal{M}\|, w) = -\infty$ otherwise. We note that in the graphical representation of the $\mathcal{M}$ presented in Figure 1, we only draw transitions with weight different from $-\infty$.*

Two wBa are called equivalent if they have the same behavior. A wBa $\mathcal{M} = (Q, wt, I, F)$ will be called normalized if the set of initial states I is a singleton, i.e., if $I = \{q_0\}$. In this case we denote it by $\mathcal{M} = (Q, wt, q_0, F)$.

Lemma 16 *For every wBa over A and K, we can effectively construct an equivalent normalized wBa over A and K.*

Sketch of the proof: Let $\mathcal{M} = (Q, wt, I, F)$ be a wBa over A and K. We define the normalized wBa $\mathcal{A} = (Q', wt', q_0, F')$ by setting $Q' = Q \cup \{q_0\} \cup S$ where q_0 is a new state, and $S = \{s_q \mid q \in Q\}$ is a set of copies of Q, $F' = F \cup \{s_q \in S \mid q \in F\}$, and $wt' : Q' \times A \times Q' \to K$ is defined in the following way:

$$\bullet wt'(q_0, a, \overline{p}) = \sum_{p' \in I, wt(p', a, \overline{p}) \neq \mathbf{1}, \mathbf{0}} wt(p', a, \overline{p}) \quad \text{if } \overline{p} \in Q,$$

$$\bullet wt'(q_0, a, s_q) = \mathbf{1} \quad \text{if } s_q \in S, \text{ and there exists } \widetilde{q} \in I, \text{ with } wt(\widetilde{q}, a, q) = \mathbf{1}$$

$$\bullet wt'(q, a, \overline{q}) = wt(q, a, \overline{q}) \quad \text{if } q, \overline{q} \in Q,$$

$$\bullet wt'(s_q, a, s_{\overline{q}}) = wt(q, a, \overline{q}) \quad \text{if } s_q, s_{\overline{q}} \in S,$$

$$\bullet wt'(p, a, \overline{p}) = \mathbf{0} \quad \text{otherwise}$$

Then, for every $w = a_0 a_1 \ldots \in A^\omega$, using the completeness axioms of the monoid we get that $(\|\mathcal{A}\|, w)$ is equal to the sum of all successful paths of $\mathcal{A}$ over w with non-zero weight, which, by the definition of wt' and the completeness axioms of the monoid, is in turn equal to the sum $\overline{\sum} + \widetilde{\sum}$ where

$$\overline{\sum} = \sum_{\substack{P_w = (q_i, a_i, q_{i+1})_{i \geq 0} \\ \in succ_{\mathcal{A}}(w) \\ q_i \in Q,\ i \geq 1 \\ weight_{\mathcal{A}}(P_w) \neq \mathbf{0}}} weight_{\mathcal{A}}(P_w), \text{ and}$$

$$\widetilde{\sum} = \sum_{\substack{P_w = (q_i, a_i, q_{i+1})_{i \geq 0} \\ \in succ_{\mathcal{A}}(w) \\ q_i \in S,\ i \geq 1 \\ weight_{\mathcal{A}}(P_w) \neq \mathbf{0}}} weight_{\mathcal{A}}(P_w).$$

For every non-zero weight successful path $P_w = (q_i, a_i, q_{i+1})_{i \geq 0}$ of $\mathcal{A}$ over w, with $q_i \in Q, i \geq 1$, we obtain, using Property 1 and the definition of wt', the following

$$weight_{\mathcal{A}}(P_w)$$

$$= Val^\omega \left(\sum_{\substack{p' \in I \\ wt(p', a_0, q_1) \neq \mathbf{0}, \mathbf{1}}} wt(p', a_0, q_1), wt'(q_1, a_1, q_2), wt'(q_2, a_2, q_3), \ldots \right)$$

$$= \sum_{p' \in I, wt(p', a_0, q_1) \neq \mathbf{0}, \mathbf{1}} Val^\omega \left(wt(p', a_0, q_1), wt(q_1, a_1, q_2), wt(q_2, a_2, q_3), \ldots \right).$$

Moreover, for every non-zero weight successful path $P_w = (q_i, a_i, q_{i+1})_{i\geq 0}$ of $\mathcal{A}$ over w with $q_i \in S$ for every $i \geq 1$, there exist $p \in I, p_i \in Q\,(i \geq 0)$ such that $q_i = s_{p_i}$ for every $i \geq 1$ and

$$\begin{aligned} weight_{\mathcal{A}}(P_w) &= Val^{\omega}(\mathbf{1}, wt(p_1, a_1, p_2), wt(p_2, a_2, p_3), \ldots) \\ &= Val^{\omega}(wt(p, a_0, p_1), wt(p_1, a_1, p_2), wt(p_2, a_2, p_3), \ldots). \end{aligned}$$

We conclude that each term of $\overline{\sum}$ and $\widetilde{\sum}$ is equal to the weight of a non-zero weight successful path of $\mathcal{M}$ over w. Also, the weight of each non-zero weight successful path P_w of $\mathcal{M}$ over w appears as a term in at least one of $\overline{\sum}$, $\widetilde{\sum}$. More precisely, if $P_w = (p_i, a_i, p_{i+1})_{i\geq 0}$ is a non-zero weight successful path of $\mathcal{M}$ over w with $wt(p_0, a_0, p_1) = \mathbf{1}$, then $weight_{\mathcal{M}}(P_w)$ equals the weight of the successful path $(q_0, a_0, s_{p_1})\left(s_{p_i}, a_i, s_{p_{i+1}}\right)_{i\geq 1}$ of $\mathcal{A}$ over w. Now, if $wt(p_0, a_0, p_1) \neq \mathbf{1}$, then the weight of the successful path $(q_0, a_0, p_1)(p_i, a_i, p_{i+1})_{i\geq 1}$ of $\mathcal{A}$ over w is equal to a finite sum with one of its terms being $weight_{\mathcal{M}}(P_w)$. We thus, by the completeness axioms of the monoid and the generalization of [22, Lemma 1(ii)] for idempotent TGP-ω-valuation monoids, can conclude that $(\|\mathcal{A}\|, w) = (\|\mathcal{M}\|, w)$ as desired.

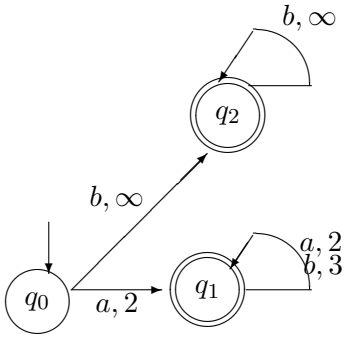

Figure 1

In all following sections, K will stand for an idempotent ordered TGP-ω-valuation monoid unless specified otherwise.

5 Describing Weighted Safety with Weighted LTL over Idempotent Ordered TGP-ω-valuation Monoids

In this section we firstly aim to introduce a notion of weighted safety for infinitary series over idempotent ordered TGP-ω-valuation monoids. More

precisely, given an idempotent ordered TGP-ω-valuation monoid K, we will define for every $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$ the notion of k-safe infinitary series. We prove that an infinitary series s over K and an alphabet A, is k-safe iff the infinitary language that contains all $w \in A^\omega$ whose coefficient on s is greater or equal to k is an infinitary safety language. We provide fragments of a weighted specification language whose formulas' semantics are infinitary series that express this notion of weighted safety. More specifically, we will extend the definition of the weighted *LTL* over idempotent ordered GP-ω-valuation monoids presented in [22] to idempotent ordered TGP-ω-valuation monoids, and determine for every $k \setminus \{\mathbf{0}, \mathbf{1}\}$ syntactic fragments of this logic such that the semantics of each formula in these fragments is a k-safe infinitary series. By presenting a counterexample we conclude that for arbitrary idempotent ordered TGP-ω-valuation monoids K, and $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, the class of semantics of formulas in the aforementioned fragments does not coincide with the class of k-safe infinitary series.

Definition 17 *Let $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$ and $s \in K\langle\langle A^\omega \rangle\rangle$. Series s will be called k-safe if for every $w \in A^\omega$ the following hold: If $\forall i > 0, \exists u \in A^\omega$ such that $(s, w_{<i}u) \geq k$, then $(s, w) \geq k$.*

Example 18 *Let $A = \{a, b\}$. We consider the infinitary series $S \in K_2\langle\langle A^\omega \rangle\rangle$ defined in the following way. For every $w \in A^\omega$, we let $(S, w) = 3^k$ if $w \in a^k b^+ A^\omega$ where $k > 0$, or $(S, w) = 3$ if $w = a^\omega$, or $(S, w) = -\infty$ otherwise. Clearly,*

$$(S, w) \geq 3 \text{ iff } w \in a^\omega \cup \left(\bigcup_{k>0} a^k b^+ A^\omega \right)$$

where $a^\omega \cup \left(\bigcup_{k>0} a^k b^+ A^\omega \right) = a^+ b^+ A^\omega \cup a^\omega = aA^\omega$. We prove that S is 3-safe. Let $w \in A^\omega$ such that $\forall i > 0, \exists u \in A^\omega$ with $(S, w_{<i}u) \geq 3$. Then, $\forall i > 0$, $w_{<i}u = aw'$ for some $w' \in A^\omega$. Then, $w(0) = a$, which implies $w \in aA^\omega$. Thus, $(S, w) \geq 3$, which implies that S is 3-safe as desired.

Example 19 *Let $A = \{a, b\}$. We consider three infinitary series $R \in K_2\langle\langle A^\omega \rangle\rangle$ defined in the following way. For every $w \in A^\omega$ we let $(R, w) = 3^{k+2}$ if $w \in a^k b^+ A^\omega$ where $k > 0$, or $(R, w) = 3$ if $w = a^\omega$, or $(R, w) = -\infty$ otherwise. Clearly,*

$$(R, w) \neq -\infty \text{ iff } w \in aA^\omega.$$

We can prove that R is 3-safe, with the same arguments we used in the previous example. We prove that R is not 4-safe. To this end, let $w = a^\omega$. Then, $\forall i > 0$, $(R, w_{<i}b^\omega) = 3^{i+1} \geq 3^2 > 4$, however $(R, w) = 3 < 4$, and this concludes our claim.

Lemma 20 *Let $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}, k' \in K$. Then, $k' \in K\langle\langle A^\omega \rangle\rangle$ is k-safe.*

Proof: If $k' < k$, then our claim trivially holds. If $k' \geq k$, then for every $w \in A^\omega$, $(k', w) \geq k$, and it holds: $\forall i > 0, \exists u \in A^\omega$ such that $(k', w_{<i}u) \geq k$, and $(k', w) \geq k$ which proves our claim. □

Theorem 21 *Let $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$ and $s \in K\langle\langle A^\omega \rangle\rangle$. Then, s is k-safe iff $L_{\geq k}(s)$ is a safety language.*

Proof: Let s be k-safe. Then, for every $w \in A^\omega$, it holds: If $\forall i > 0, \exists u \in A^\omega$ such that $(s, w_{<i}u) \geq k$, then $(s, w) \geq k$. This implies that for every $w \in A^\omega$, it holds: If $\forall i > 0, \exists u \in A^\omega$ such that $w_{<i}u \in L_{\geq k}(s)$, then $w \in L_{\geq k}(s)$, which in turn implies that $L_{\geq k}(s)$ is a safety language.

Now, we let $L_{\geq k}(s)$ be a safety language. Then, for every $w \in A^\omega$, it holds: If $\forall i > 0, \exists u \in A^\omega$ such that $w_{<i}u \in L_{\geq k}(s)$, then $w \in L_{\geq k}(s)$. Thus, for every $w \in A^\omega$, it holds: If $\forall i > 0, \exists u \in A^\omega$ such that $(s, w_{<i}u) \geq k$, then $(s, w) \geq k$, and so s is k-safe as desired. □

Remark 22 *We can obtain a to Definition 17 equivalent definition for k-safe infinitary series by considering the contrapositive condition. More specifically, for $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, $s \in K\langle\langle A^\omega \rangle\rangle$ is k-safe if for every $w \in A^\omega$ it holds: if $(s, w) < k$, then there exists an $i > 0$ such that for all $u \in A^\omega$, $(s, w_{<i}u) < k$. The equivalence of the two definitions can be obtained in a straightforward way using Theorem* 21.

We recall now from [22] that the syntax of the weighted LTL over AP and K is given by the grammar

$$\varphi ::= k \mid a \mid \neg a \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \bigcirc\varphi \mid \varphi U \varphi \mid \Box\varphi$$

where $k \in K$ and $a \in AP$. As usual we identify $\neg\neg a$ with a for every $a \in AP$. We shall denote by $LTL(K, AP)$ the class of all weighted LTL formulas over AP and K. We note that formulas in $LTL(K, AP)$ are by construction in Negation Normal Form, i.e., negation is applied only to atomic propositions.

Definition 23 *The semantics* $\|\varphi\|$ *of formulas* $\varphi \in LTL(K, AP)$ *are represented as infinitary series in* $K\langle\langle(\mathcal{P}(AP))^{\omega}\rangle\rangle$ *inductively defined in the following way.*

- $(\|k\|, w) = k$
- $(\|\varphi \wedge \psi\|, w) = (\|\varphi\|, w) \cdot (\|\psi\|, w)$
- $(\|a\|, w) = \begin{cases} \mathbf{1} & \text{if } a \in w(0) \\ \mathbf{0} & \text{otherwise} \end{cases}$
- $(\|\varphi \vee \psi\|, w) = (\|\varphi\|, w) + (\|\psi\|, w)$
- $(\|\neg a\|, w) = \begin{cases} \mathbf{1} & \text{if } a \notin w(0) \\ \mathbf{0} & \text{otherwise} \end{cases}$
- $(\|\bigcirc\varphi\|, w) = (\|\varphi\|, w_{\geq 1})$
- $(\|\varphi U \psi\|, w)$
  $= \sum_{i \geq 0} Val^{\omega}((\|\varphi\|, w_{\geq 0}), \ldots, (\|\varphi\|, w_{\geq i-1}), (\|\psi\|, w_{\geq i}), \mathbf{1}, \mathbf{1}, \ldots)$
- $(\|\Box\varphi\|, w) = Val^{\omega}((\|\varphi\|, w_{\geq i}))_{i \geq 0}$

We shall denote by *true* the formula $\mathbf{1} \in K$. Two formulas $\varphi, \psi \in LTL(K, AP)$ will be called equivalent, and we will denote this by $\varphi \equiv \psi$, iff $(\|\varphi\|, w) = (\|\psi\|, w)$ for every $w \in (\mathcal{P}(AP))^{\omega}$. We define the *weak until* operator $\widetilde{U}$ by letting $\varphi \widetilde{U} \psi := \Box\varphi \vee (\varphi U \psi)$. The syntactic boolean fragment $bLTL(K, AP)$ of $LTL(K, AP)$ is given by the grammar

$$\varphi ::= \mathbf{0} \mid true \mid a \mid \neg a \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \bigcirc\varphi \mid \varphi U \varphi \mid \Box\varphi$$

where $a \in AP$. Inductively, we can prove that $Im(\|\varphi\|) \subseteq \{\mathbf{0}, \mathbf{1}\}$ for every $\varphi \in bLTL(K, AP)$, and the semantics of the formulas in $bLTL(K, AP)$ and corresponding classical *LTL* formulas coincide. The syntactic fragment $sbLTL(K, AP)$ of $bLTL(K, AP)$ is given by the grammar

$$\varphi ::= true \mid a \mid \neg a \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \bigcirc\varphi \mid \varphi \widetilde{U} \varphi \mid \Box\varphi.$$

We observe that for every $\varphi \in sbLTL(K, AP)$ it holds φ in $sLTL(AP)$, and for every $w \in (\mathcal{P}(AP))^{\omega}$, $(\|\varphi\|, w) = \mathbf{1}$ iff $w \models \varphi$. We will call every formula in $sbLTL(K, AP)$ a boolean safety formula.

As in [22] we let a restricted *LTL-step formula* be an $LTL(K, AP)$-formula of the form $\bigvee_{1 \leq i \leq n} (k_i \wedge \varphi_i)$ with $k_i \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, $\varphi_i \in bLTL(K, AP)$ for all $1 \leq i \leq n$. We denote by *r-stLTL*(K, AP) the class of all restricted *LTL*-step formulas over AP and K. We now let $L_k = \{k' \in K \mid k' \geq k\}$ for every $k \in K$. For a given $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, we let a *k-LTL-step formula* be an

$LTL(K,AP)$-formula of the form $\bigvee_{1\leq i\leq n}(k_i \wedge \varphi_i)$ with $k_i \in L_k \setminus \{\mathbf{0},\mathbf{1}\}$ and $\varphi_i \in sbLTL(K,AP)$ for all $1 \leq i \leq n$. We denote by $k\text{-}stLTL(K,AP)$ the class of all k-LTL-step formulas over AP and K.

Example 24 *Let* $AP = \{a,b\}$, $\varphi = \Box((a \wedge 2) \vee (b \wedge 3)) \in LTL(K_3,AP)$. *Then, for every* $w \in (\mathcal{P}(AP))^\omega$ *the following hold:* $(\|\varphi\|, w) = -\infty$, *if for some position* $i \geq 0$, *both* a, b *do not belong to* $w(i)$. *Moreover,* $(\|\varphi\|, w) = 3$, *if for every position* $i \geq 0$, *at least one of* a, *and* b *belongs to* $w(i)$, *and for at least one* $i \geq 0$, b *belongs to* $w(i)$. *Finally,* $(\|\varphi\|, w) = 2$, *if for every position* $i \geq 0$, a *belongs to* $w(i)$, *but* b *does not.*

Example 25 *Let* $AP = \{a,b\}$, $\varphi = \Box((a \wedge 2) \vee (b \wedge 3)) \in LTL(K_2,AP)$. *Then, for every* $w \in (\mathcal{P}(AP))^\omega$ *the following hold:* $(\|\varphi\|, w) = -\infty$, *if for some position* $i \geq 0$, *both* a, b *do not belong to* $w(i)$. *Moreover,* $(\|\varphi\|, w) = 3$, *if for every position* $i \geq 0$, $a \in w(i)$, *or* $b \in w(i)$, *and there exist infinitely many* $i \geq 0$, *such that* $b \in w(i)$. *Finally,* $(\|\varphi\|, w) = 2$, *if for every position* $i \geq 0$, $a \in w(i)$, *or* $b \in w(i)$, *and there exist only finitely many* $i \geq 0$, *such that* $b \in w(i)$.

Remark 26 *Let* $k \in K \setminus \{\mathbf{0},\mathbf{1}\}$, *and* $\varphi \in k\text{-}stLTL(K,AP)$. *Then,* $\varphi = \bigvee_{1\leq i\leq n}(k_i \wedge \varphi_i)$ *where* $k_i \in L_k \setminus \{\mathbf{0},\mathbf{1}\}$, *and* $\varphi_i \in sbLTL(K,AP)$ *for every* $1 \leq i \leq n$. *Then,*

$$Im(\| \varphi \|) \subseteq \{\mathbf{0}\} \cup \left\{ \sum_{1\leq j\leq l, i_j \in \{1,\ldots,n\}} k_{i_j} \;\middle|\; l \in \{1,\ldots,n\} \right\}.$$

Clearly, $Im(\| \varphi \|)$ *is finite. Moreover, it holds* $k_i \geq k$ *for every* $i \in \{1,\ldots,n\}$, *i.e.,* $k_i = k + k_i$ *for every* $i \in \{1,\ldots,n\}$. *Thus, for every* $l \in \{1,\ldots,n\}$,

$$\begin{aligned} \sum_{1\leq j\leq l, i_j \in \{1,\ldots,n\}} k_{i_j} &= k_{i_1} + \sum_{2\leq j\leq l, i_j \in \{1,\ldots,n\}} k_{i_j} \\ &= k + k_{i_1} + \sum_{2\leq j\leq l, i_j \in \{1,\ldots,n\}} k_{i_j} \\ &= k + \sum_{1\leq j\leq l, i_j \in \{1,\ldots,n\}} k_{i_j} \geq k, \end{aligned}$$

where the first and last equality hold by the completeness axioms of the monoid, and the second equality and the last inequality hold by the definition of the natural order induced by idempotency. Clearly, for all $w \in (\mathcal{P}(AP))^{\omega}$ *it holds* $(\|\varphi\|, w) < k \Leftrightarrow (\|\varphi\|, w) = \mathbf{0}$.

We recall now from [22] the definition of totally restricted U-nesting LTL-formulas over AP and K. Due to the restricted version of distributivity of Val^{ω} over finite sums that holds in TGP-ω-valuation monoids, additional syntactic restrictions must be imposed to the formulas of the fragment of totally restricted U-nesting LTL-formulas in order to establish the effective translation of formulas of the weighted LTL over TGP-ω-valuation monoids to wBa. To that end, we introduce the fragment of $\vee$-*totally restricted* U-*nesting* LTL-formulas over AP and K, which is defined inductively exactly as the fragment of totally restricted U-nesting LTL-formulas with the only difference that we require a stronger syntactic restriction in the inductive step that refers to the $\vee$-operator.

Definition 27 *The fragment of totally restricted* U*-nesting LTL-formulas over* AP *and* K*, denoted by* $t\text{-}RULTL(K, AP)$*, is the least class of formulas in* $LTL(K, AP)$ *which is defined inductively in the following way.*

1. $k \in t\text{-}RULTL(K, AP)$ *for every* $k \in K$,
2. $bLTL(K, AP) \subseteq t\text{-}RULTL(K, AP)$,
3. $r\text{-}stLTL(K, AP) \subseteq t\text{-}RULTL(K, AP)$,
4. *If* $\varphi \in t\text{-}RULTL(K, AP)$, *then* $\bigcirc\varphi \in t\text{-}RULTL(K, AP)$,
5. *If* $\varphi, \psi \in t\text{-}RULTL(K, AP)$, *then* $\varphi \vee \psi \in t\text{-}RULTL(K, AP)$,
6. *If* $\varphi \in bLTL(K, AP)$, *and* $\psi \in r\text{-}stLTL(K, AP)$, *or* $\psi = \xi U \lambda$, *or* $\psi = \Box\xi$ *with* $\xi, \lambda \in r\text{-}stLTL(K, AP)$, *then* $\varphi \wedge \psi, \psi \wedge \varphi \in t\text{-}RULTL(K, AP)$,
7. *If* $\varphi, \psi \in r\text{-}stLTL(K, AP)$, *then* $\varphi U \psi \in t\text{-}RULTL(K, AP)$,
8. *If* $\varphi \in r\text{-}stLTL(K, AP)$, *then* $\Box\varphi \in t\text{-}RULTL(K, AP)$.

The fragment of $\vee$-totally restricted U-nesting LTL-formulas over AP and K, denoted by $\vee$-t-$RULTL(K, AP)$, is the least class of formulas in $LTL(K, AP)$ which is defined inductively exactly as the fragment of totally restricted U-nesting LTL-formulas over AP and K with the only difference that the inductive step that refers to the $\vee$-operator (condition 5) is now defined by the following statement

- *If $\varphi, \psi \in r\text{-}stLTL(K, AP)$, or $\varphi = \lambda U \xi, \psi = \square\xi$ with $\lambda, \xi \in r\text{-}stLTL(K, AP)$, then $\varphi \vee \psi, \psi \vee \varphi \in \vee$-$t$-$RULTL(K, AP)$.*

Example 28 *Let $\varphi = \square(a \wedge 2) \wedge (trueUc) \in t\text{-}RULTL(K_1, AP)$ where $AP = \{a, b, c\}$. Then, for every $w \in (\mathcal{P}(AP))^\omega$ the following hold: $(\|\varphi\|, w) = 2$, if $\exists i \geq 0$, such that $c \in w(i)$, and $\forall j \geq 0$, $a \in w(i)$. Otherwise, $(\|\varphi\|, w) = -\infty$.*

Theorem 29 ([22]) *Let K be an idempotent ordered GP-ω-valuation monoid, and $\varphi \in t\text{-}RULTL(K, AP)$. Then, we can effectively construct a wBa over $\mathcal{P}(AP)$ and K recognizing $\|\varphi\|$.*

Following the proofs of [22] we generalize the above theorem for idempotent ordered TGP-ω-valuation monoids and $\vee$-totally restricted U-nesting LTL-formulas. A sketch of the proof can be found in the Appendix.

Theorem 30 *Let K be an idempotent ordered TGP-ω-valuation monoid, and $\varphi \in \vee$-t-$RULTL(K, AP)$. Then, we can effectively construct a wBa over $\mathcal{P}(AP)$ and K recognizing $\|\varphi\|$.*

We aim to introduce syntactic fragments of the weighted LTL over K, and AP with the property that for specific $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$ the semantics of the formulas in the fragments are k-safe infinitary series over K. We also aim that the formulas in these fragments preserve the property of totally restricted U-nesting LTL formulas, and $\vee$-totally restricted U-nesting LTL formulas that can be effectively translated to an equivalent wBa. To motivate the restrictions that we will impose to these fragments we present the following examples of $\vee$-t-$RULTL(K, AP)$-formulas whose semantics are not k-safe for specific $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$.

Example 31 *We let $AP = \{a, b\}$. Then, $\varphi = (3 \wedge a)\, U\, (3 \wedge b) \in \vee$-$t$-$RULTL(K_2, AP)$. It holds*

$$(\| \varphi \|, w) = \begin{cases} 3 & if\ w \in \{\{a\}, \{a, b\}\}^* \{\{b\}, \{a, b\}\} (\mathcal{P}(AP))^\omega \\ -\infty & otherwise \end{cases}$$

Let $w = \{a\}^\omega$, then for every $i > 0$, $(\| \varphi \|, w_{<i}\{b\}^\omega) = 3 \geq 3$. However, $(\| \varphi \|, \{a\}^\omega) = -\infty < 3$, which implies that $\| \varphi \|$ is not 3-safe.

Let now $\psi = \Box(2 \wedge a) \vee ((2 \wedge a)\, U\, (3 \wedge b)) \in \vee$-t-RULTL$(K_2, AP)$. It holds

$$(\| \psi \|, w) = \begin{cases} 3 & if\ w \in \{\{a\}, \{a,b\}\}^* \{\{b\}, \{a,b\}\} (\mathcal{P}(AP))^\omega \\ 2 & if\ w = \{a\}^\omega \\ -\infty & otherwise \end{cases}$$

Let $w = \{a\}^\omega$, then for every $i > 0$, $(\| \psi \|, w_{<i}\{b\}^\omega) = 3 \geq 3$. However, $(\| \psi \|, \{a\}^\omega) = 2 < 3$, which implies that $\| \psi \|$ is not 3-safe.

Definition 32 *Let $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$. The fragment of k-safe totally restricted U-nesting LTL-formulas over AP and K, denoted by k-t-RULTL(K, AP), is the least class of formulas in LTL(K, AP) which is defined inductively in the following way.*

1. $L_k \subseteq k$-t-RULTL(K, AP),
2. $sbLTL(K, AP) \subseteq k$-t-RULTL(K, AP),
3. k-stLTL$(K, AP) \subseteq k$-t-RULTL(K, AP),
4. *If* $\varphi \in k$-t-RULTL(K, AP), *then* $\bigcirc\varphi \in k$-t-RULTL(K, AP),
5. *If* $\varphi, \psi \in k$-t-RULTL(K, AP), *then* $\varphi \vee \psi \in k$-t-RULTL(K, AP),
6. *If* $\varphi \in sbLTL(K, AP)$, *and* $\psi \in k$-stLTL(K, AP) *or* $\psi = \xi\widetilde{U}\lambda$, *or* $\psi = \Box\xi$ *with* $\xi, \lambda \in k$-stLTL(K, AP), *then* $\varphi \wedge \psi, \psi \wedge \varphi \in k$-t-RULTL$(K, AP)$,
7. *If* $\varphi, \psi \in k$-stLTL(K, AP), *then* $\varphi\widetilde{U}\psi \in k$-t-RULTL$(K, AP)$,
8. *If* $\varphi \in k$-stLTL(K, AP), *then* $\Box\varphi \in k$-t-RULTL(K, AP).

The fragment of k-safe $\vee$-totally restricted U-nesting LTL-formulas over AP and K, which is denoted by k-$\vee$-t-RULTL(K, AP), is the least class of formulas in LTL(K, AP) which is defined inductively exactly as the fragment of k-safe totally restricted U-nesting LTL-formulas over AP and K with the only difference that the inductive step that refers to the $\vee$-operator (condition 5) is now defined by the following statement

· *If* $\varphi, \psi \in k\text{-}stLTL(K, AP)$, *then* $\varphi \vee \psi \in k\text{-}\vee\text{-}t\text{-}RULTL(K, AP)$.

Remark 33 *We note that* $k\text{-}\vee\text{-}t\text{-}RULTL(K, AP) \subseteq k\text{-}t\text{-}RULTL(K, AP)$. *Moreover, for all* $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, $\mathbf{0} \notin L_k$, *and for every* $k\text{-}t\text{-}RULTL(K, AP)$*-formula (resp. for every* $k\text{-}\vee\text{-}t\text{-}RULTL(K, AP)$*-formula), we can construct, using the definition of* $\widetilde{U}$, *an equivalent* $t\text{-}RULTL(K, AP)$*-formula (resp. an equivalent* $\vee\text{-}t\text{-}RULTL(K, AP)$*-formula).*

Next, we will prove that the semantics of $k\text{-}t\text{-}RULTL(K, AP)$-formulas are k-safe infinitary series over K and $\mathcal{P}(AP)$. We shall need the following lemmas.

Lemma 34 *If* $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, *and* $\varphi \in sbLTL(K, AP)$, *then* $\|\varphi\|$ *is* k*-safe.*

Proof: Let $w \in (\mathcal{P}(AP))^\omega$ such that $\forall i > 0, \exists u \in (\mathcal{P}(AP))^\omega$, with $(\|\varphi\|, w_{<i}u) \geq k$. By Property 4 of TGP-$\omega$-valuation monoids, the facts that $k \neq \mathbf{0}$, and $Im(\|\varphi\|) = \{\mathbf{0}, \mathbf{1}\}$, we get that $\forall i > 0, \exists u \in (\mathcal{P}(AP))^\omega$, such that $(\|\varphi\|, w_{<i}u) = \mathbf{1}$, i.e., $\forall i > 0, \exists u \in (\mathcal{P}(AP))^\omega$, with $w_{<i}u \models \varphi$. Then, since $\mathcal{L}(\phi)$ is a safety language, $w \models \varphi$, which implies that $(\|\varphi\|, w) = \mathbf{1} \geq k$ as desired. □

Let now $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, and $\varphi = \bigvee_{1 \leq i \leq n} (k_i \wedge \varphi_i) \in k\text{-}stLTL(K, AP)$. We assign to φ the formula $\varphi_b := \bigvee_{1 \leq i \leq n} \varphi_i \in sbLTL(K, AP)$. In addition, if $\varphi = \psi \widetilde{U} \xi$ (resp. $\varphi = \Box\psi, \varphi = \psi U \xi$), with $\psi, \xi \in k\text{-}stLTL(K, AP)$, we assign to φ, the formula $\varphi_b := \psi_b \widetilde{U} \xi_b \in sbLTL(K, AP)$ (resp. $\varphi_b := \Box\psi_b \in sbLTL(K, AP)$, $\varphi_b := \psi_b U \xi_b \in bLTL(K, AP)$). In fact, the operator φ_b assigns to φ a boolean formula whose language coincides with $L_{\geq k}(\|\varphi\|)$. With Lemmas 35, 36, that follow, we prove the latter property.

Lemma 35 *Let* $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, *and* $\varphi \in k\text{-}stLTL(K, AP)$. *Then, the following is true: For every* $w \in (\mathcal{P}(AP))^\omega$, $(\|\varphi\|, w) \geq k$ *iff* $w \models \varphi_b$.

Proof: Let $\varphi = \bigvee_{1 \leq i \leq n} (k_i \wedge \varphi_i) \in k\text{-}stLTL(K, AP)$. Then, for every $w \in (\mathcal{P}(AP))^\omega$ it holds:

$$(\|\varphi\|, w) \geq k$$
iff

$$\left(\left\|\bigvee_{1\leq i\leq n}(k_i\wedge\varphi_i)\right\|,w\right)\geq k$$
iff
$$\left(\left\|\bigvee_{1\leq i\leq n}(k_i\wedge\varphi_i)\right\|,w\right)\neq \mathbf{0}$$
iff
$$\exists i\in\{1,\ldots,n\} \text{ s.t. } (\|\varphi_i\|,w)=\mathbf{1}$$
iff
$$w\models \bigvee_{1\leq i\leq n}\varphi_i=\varphi_b,$$

where the second equivalence is obtained by Remark 26. The straight implication of the third equivalence is obtained by assuming the contrary and by the semantics of $LTL(K,AP)$-formulas. The inverse implication of the third equivalence holds by the semantics of $LTL(K,AP)$-formulas and the fact that K is zero-sum free. The last equivalence is obtained by the coincidence of the semantics of $sbLTL(K,AP)$-formulas with the corresponding $sLTL(AP)$-formulas. □

Lemma 36 *Let $\varphi=\psi\widetilde{U}\xi$, or $\varphi=\Box\psi$, or $\varphi=\psi U\xi$ with $\psi,\xi\in k$-$stLTL(K,AP)$. Then, the following is true: For every $w\in(\mathcal{P}(AP))^{\omega}$, $(\|\varphi\|,w)\geq k$ iff $w\models\varphi_b$.*

Proof: Let $\varphi=\Box\psi$, with $\psi\in k$-$stLTL(K,AP)$. Then, the following equivalences are true:

$$(\|\varphi\|,w)=(\|\Box\psi\|,w)=Val^{\omega}\left((\|\psi\|,w_{\geq j})\right)_{j\geq 0}\geq k$$
$$\text{iff } (\|\psi\|,w_{\geq j})\geq k,\forall j\geq 0$$
iff
$$w_{\geq j}\models\psi_b,\forall j\geq 0$$
iff
$$w\models\Box\psi_b=\varphi_b.$$

To derive the straight implication of the first equivalence we assume the contrary. Then, since K is ordered, $\exists j \geq 0$ with $(\|\psi\|, w_{\geq j}) < k$, which by Remark 26 implies that $(\|\psi\|, w_{\geq j}) = \mathbf{0}$. Thus, by definition of Val^{ω}, we get $Val^{\omega}((\|\psi\|, w_{\geq j}))_{j\geq 0} = \mathbf{0}$, which is a contradiction. The inverse implication of the first equivalence is concluded by Property 5. The second equivalence is concluded by Lemma 35, and the last one by the semantics of $LTL(AP)$-formulas.

Let $\varphi = \psi U \xi$ with $\psi, \xi \in k\text{-}stLTL(K, AP)$, and $w \in (\mathcal{P}(AP))^{\omega}$. Then, the following equivalences are true:

$$(\|\varphi\|, w) = (\|\psi U \xi\|, w)$$
$$= \sum_{h\geq 0} Val^{\omega}((\|\psi\|, w_{\geq 0}), \ldots, (\|\psi\|, w_{\geq h-1}), (\|\xi\|, w_{\geq h}), \mathbf{1}, \mathbf{1}, \ldots) \geq k$$
$$\text{iff}$$
$$\exists h \geq 0 \text{ such that } (\|\xi\|, w_{\geq h}) \geq k \text{ and } (\|\psi\|, w_{\geq l}) \geq k, 0 \leq l \leq h-1$$
$$\text{iff}$$
$$\exists h \geq 0 \text{ such that } w_{\geq h} \models \xi_b \text{ and } w_{\geq l} \models \psi_b, 0 \leq l \leq h-1$$
$$\text{iff}$$
$$w \models \psi_b U \xi_b = \varphi_b.$$

In order to prove the straight implication of first equivalence we assume the contrary. Then, since K is ordered, we get that for all $h \geq 0$ it holds

$$(\|\xi\|, w_{\geq h}) < k, \text{ or } (\|\psi\|, w_{\geq l}) < k \text{ for some } l \in \{0, \ldots, h-1\}.$$

This, by Remark 26, implies that for all $h \geq 0$ it holds

$$(\|\xi\|, w_{\geq h}) = \mathbf{0}, \text{ or } (\|\psi\|, w_{\geq l}) = \mathbf{0} \text{ for some } l \in \{0, \ldots, h-1\}.$$

Thus,

$$\forall h \geq 0, Val^{\omega}((\|\psi\|, w_{\geq 0}), \ldots, (\|\psi\|, w_{\geq h-1}), (\|\xi\|, w_{\geq h}), \mathbf{1}, \mathbf{1}, \ldots) = \mathbf{0},$$

and so by the generalization of [22, Lemma 1(ii)] for idempotent TGP-ω-valuation monoids it holds:

$$\sum_{h\geq 0} Val^{\omega}((\|\psi\|, w_{\geq 0}), \ldots, (\|\psi\|, w_{\geq h-1}), (\|\xi\|, w_{\geq h}), \mathbf{1}, \mathbf{1}, \ldots) = \mathbf{0}$$

which is a contradiction. The inverse implication of the first equivalence is derived by the completeness axioms of the monoid and Properties 4, 5. As in the previous case, the second equivalence is concluded by Lemma 35, and the last one by the semantics of $LTL(AP)$-formulas.

Finally, if $\varphi = \psi \widetilde{U} \xi$, $\psi, \xi \in k\text{-}stLTL(K, AP)$, and $w \in (\mathcal{P}(AP))^{\omega}$, the following equivalences are true:

$$(\|\varphi\|, w) = \left(\left\|\psi \widetilde{U} \xi\right\|, w\right) = (\|\Box\psi\|, w) + (\|\psi U \xi\|, w) \geq k$$
$$\text{iff}$$
$$(\|\Box\psi\|, w) \geq k \text{ or } (\|\psi U \xi\|, w) \geq k$$
$$\text{iff}$$
$$w \models \Box\psi_b \text{ or } w \models \psi_b U \xi_b$$
$$\text{iff}$$
$$w \models \psi_b \widetilde{U} \xi_b = \varphi_b.$$

The straight implication of the first equivalence is derived by assuming the contrary and Lemma 8, while the inverse implication of the first equivalence is derived by the definition of the natural order induced by idempotency, the associativity, and the commutativity of the monoid. The third equivalence is derived by the previous cases, and the last one by the semantics of $LTL(AP)$-formulas. □

Lemma 37 *Let $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, and $\varphi \in k\text{-}stLTL(K, AP)$. Then, φ is k-safe.*

Proof: Let $\varphi = \bigvee_{1 \leq i \leq n} (k_i \wedge \varphi_i) \in k\text{-}stLTL(K, AP)$, and $w \in (\mathcal{P}(AP))^{\omega}$ such that for every $j > 0, \exists u_j \in (\mathcal{P}(AP))^{\omega}$ with $(\|\varphi\|, w_{<j} u_j) \geq k$. Then, by Lemma 35 we get that $\forall j > 0, \exists u_j \in (\mathcal{P}(AP))^{\omega}$ with $w_{<j} u_j \models \varphi_b = \bigvee_{1 \leq i \leq n} \varphi_i$. Then, for at least one $i \in \{1, \ldots, n\}$, there exist infinitely many $j > 0$, such that $w_{<j} u_j \models \varphi_i \in sbLTL(K, AP)$ for some $u_j \in (\mathcal{P}(AP))^{\omega}$. Let now $j_1 < j_2 < j_3 < \ldots$ be the sequence of positions with $w_{<j_l} u_{j_l} \models \varphi_i$ for some $u_{j_l} \in (\mathcal{P}(AP))^{\omega}$, $l \geq 1$. Let $j < j_h$ for some $h \geq 1$, then $w_{<j_h} = w_{<j} w(j) \ldots w(j_h - 1)$. Thus, for $u_j = w(j) \ldots w(j_h - 1) u_{j_h}$ we get $w_{<j} u_j \models \varphi_i$ as desired. Since for all $j > 0$, there exist $h > 0$ such that $j \leq j_h$, we conclude that for all $j > 0$, there exists $u \in (\mathcal{P}(AP))^{\omega}$ with $w_{<j} u \models \varphi_i$. Since $\varphi_i \in sLTL(AP)$, we conclude that $w \models \varphi_i$. This implies

that $(\|\varphi_i\|, w) = \mathbf{1}$, and hence $(\|\varphi\|, w) = \left(\left\|\bigvee_{1\leq i\leq n}(k_i \wedge \varphi_i)\right\|, w\right) \geq k$ as desired. □

We are ready now to prove that the semantics of k-t-$RULTL(K, AP)$-formulas are k-safe infinitary series.

Theorem 38 *Let $k \in K\setminus\{\mathbf{0},\mathbf{1}\}$, and $\varphi \in k$-t-$RULTL(K, AP)$. Then, $\|\varphi\| \in K\langle\langle(\mathcal{P}(AP))^{\omega}\rangle\rangle$ is k-safe.*

Proof: We prove the Lemma's claim by induction on the structure of k-t-$RULTL(K, AP)$-formulas. For $\varphi = k' \in L_k$, by Lemma 20 we get that $\|k'\| = k'$ is k-safe. For $\varphi \in sbLTL(K, AP)$, $\varphi \in k$-$stLTL(K, AP)$ we obtain our claim by the Lemmas 34 and 37 respectively.

Assume that $\xi, \psi \in k$-t-$RULTL(K, AP)$ such that the induction hypothesis holds for ψ, ξ. Let $\varphi = \xi \vee \psi$, and $w \in (\mathcal{P}(AP))^{\omega}$ such that $\forall i > 0$, $\exists u \in (\mathcal{P}(AP))^{\omega}$ with $(\|\varphi\|, w_{<i}u) \geq k$. Then, $\forall i > 0, \exists u \in (\mathcal{P}(AP))^{\omega}$ with $(\|\xi\|, w_{<i}u) + (\|\psi\|, w_{<i}u) \geq k$. By assuming the contrary, and using Lemma 8, we get that there exist infinitely many $i > 0$ such that $(\|\psi\|, w_{<i}u) \geq k$ for some $u \in (\mathcal{P}(AP))^{\omega}$, or there exist infinitely many $i > 0$ such that $(\|\xi\|, w_{<i}u) \geq k$ for some $u \in (\mathcal{P}(AP))^{\omega}$. Assume now that the first case is true and let $i_1 < i_2 < i_3 < \ldots$ be the sequence of positions with $\left(\|\psi\|, w_{<i_j}u_{i_j}\right) \geq k$ for some $u_{i_j} \in (\mathcal{P}(AP))^{\omega}$, $j > 0$. Let $i < i_h$ for some $h \geq 1$, then $w_{<i_h} = w_{<i}w(i)\ldots w(i_h - 1)$. Thus, for $u = w(i)\ldots w(i_h - 1)\,u_{i_h}$ we get $(\|\psi\|, w_{<i}u) \geq k$ as desired. Since for all $i > 0$, there exist $h > 0$ such that $i < i_h$, we conclude that for all $i > 0$, there exists $u \in (\mathcal{P}(AP))^{\omega}$ with $(\|\psi\|, w_{<i}u) \geq k$. Then, by induction hypothesis ψ is k-safe, i.e., $(\|\psi\|, w) \geq k$, which implies that $(\|\psi\|, w) = k + (\|\psi\|, w)$. Thus, $(\|\varphi\|, w) = (\|\xi\|, w) + (\|\psi\|, w) = k + ((\|\xi\|, w) + (\|\psi\|, w)) = k + (\|\varphi\|, w)$, where the second equality is derived by the definition of the natural order induced by idempotency, the associativity, and the commutativity of the monoid. Thus, $(\|\varphi\|, w) \geq k$ as desired. Now, if the second case is true, we conclude our claim with the same arguments.

Next, we let $\varphi = \bigcirc\xi$, and $w \in (\mathcal{P}(AP))^{\omega}$ such that $\forall i > 0, \exists u \in (\mathcal{P}(AP))^{\omega}$ with $(\|\varphi\|, w_{<i}u) = \left(\|\xi\|, (w_{<i}u)_{\geq 1}\right) \geq k$. This implies that for every $i > 0$, $\exists u \in (\mathcal{P}(AP))^{\omega}$ with $\left(\|\xi\|, (w_{\geq 1})_{<i}\,u\right) \geq k$ (observe that for all $i > 0$, $(w_{\geq 1})_{<i}\,u = (w_{<i+1}u)_{\geq 1}$), and since by induction hypothesis ξ is k-safe, we get $(\|\xi\|, w_{\geq 1}) \geq k$, i.e., $(\|\bigcirc\xi\|, w) \geq k$ as desired.

Let now $\varphi = \psi \wedge \xi$ with $\psi \in sbLTL(K, AP)$, and $\xi \in k$-$stLTL(K, AP)$, or $\xi = \zeta\widetilde{U}\lambda$, or $\xi = \Box\zeta$ with $\zeta, \lambda \in k$-$stLTL(K, AP)$, such that the induction

hypothesis holds for ψ, ξ. Let $w \in (\mathcal{P}(AP))^{\omega}$ such that $\forall i \geq 0$, $\exists u \in (\mathcal{P}(AP))^{\omega}$ with $(\|\varphi\|, w_{<i}u) \geq k$. Then, $\forall i \geq 0, \exists u \in (\mathcal{P}(AP))^{\omega}$ with $(\|\psi\|, w_{<i}u) \cdot (\|\xi\|, w_{<i}u) \geq k$, and since $k \neq \mathbf{0}$ and ψ is boolean, we get that $\forall i \geq 0, \exists u \in (\mathcal{P}(AP))^{\omega}$ with $(\|\psi\|, w_{<i}u) = \mathbf{1}$, and $(\|\psi\|, w_{<i}u) \cdot (\|\xi\|, w_{<i}u) = (\|\xi\|, w_{<i}u) \geq k$. Thus, since by Property 4 we get that $\mathbf{1}$ is the maximal element of K, it holds that $\forall i \geq 0, \exists u \in (\mathcal{P}(AP))^{\omega}$ with $(\|\psi\|, w_{<i}u) \geq k$, and $(\|\xi\|, w_{<i}u) \geq k$. Then, by induction hypothesis we get $(\|\psi\|, w) \geq k$, and $(\|\xi\|, w) \geq k$, which, taking into account that ψ is boolean and $k \neq \mathbf{0}$, implies that $(\|\psi\|, w) = \mathbf{1}$ and $(\|\xi\|, w) \geq k$, and thus $(\|\varphi\|, w) = (\|\psi\|, w) \cdot (\|\xi\|, w) \geq k$ as desired.

Let $\varphi = \psi \widetilde{U} \xi$ with $\psi, \xi \in k\text{-}stLTL(K, AP)$, and $w \in (\mathcal{P}(AP))^{\omega}$ such that $\forall i > 0, \exists u \in (\mathcal{P}(AP))^{\omega}$ such that $(\|\varphi\|, w_{<i}u) \geq k$. Then, by Lemma 36, we get that for all $i > 0$, $\exists u \in (\mathcal{P}(AP))^{\omega}$ with $w_{<i}u \models \varphi_b$, and since $\varphi_b \in sLTL(AP)$, we conclude that $w \models \varphi_b$. By Lemma 36, this implies that $(\|\varphi\|, w) \geq k$ as desired.

We now let $\varphi = \Box\psi$ with $\psi \in k\text{-}stLTL(K, AP)$, and $w \in (\mathcal{P}(AP))^{\omega}$ such that $\forall i > 0, \exists u \in (\mathcal{P}(AP))^{\omega}$ with $(\|\varphi\|, w_{<i}u) \geq k$. Then, by Lemma 36, for all $i > 0, \exists u \in (\mathcal{P}(AP))^{\omega}$, $w_{<i}u \models \varphi_b \in sLTL(AP)$. Hence, $w \models \varphi_b$, and again by Lemma 36 we get that $(\|\varphi\|, w) \geq k$ as desired. □

Corollary 39 *Let $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, and $\varphi \in k\text{-}\vee\text{-}t\text{-}RULTL(K, AP)$. Then, $\|\varphi\| \in K\langle\langle(\mathcal{P}(AP))^{\omega}\rangle\rangle$ is k-safe.*

Remark 40 *Let $k \in K_2 \setminus \{-\infty, \infty\}$. Then, by induction on the structure of formulas in $k\text{-}t\text{-}RULTL(K_2, AP)$, and using Remark 26, we can prove that $Im(||\varphi||)$ is finite, for every $\varphi \in k\text{-}t\text{-}RULTL(K_2, AP)$. In Example 18, we have presented the 2-safe infinitary series $S \in K_2\langle\langle A^{\omega}\rangle\rangle$, where $A = \{a, b\}$, with $Im(S)$ being an infinite set. We set $AP = \{a, b\}$. Clearly, $S \in K_2\langle\langle(\mathcal{P}(AP))^{\omega}\rangle\rangle$. This implies that the class of semantics of $2\text{-}t\text{-}RULTL(K_2, AP)$-formulas is a proper subclass of the class of 2-safe infinitary series over K_2, and $\mathcal{P}(AP)$.*

Corollary 41 *Let AP be a finite set of atomic propositions. Then, for the arbitrary idempotent ordered TGP-ω-valuation monoid K, and $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$ the class of semantics of $k\text{-}t\text{-}RULTL(K, AP)$-formulas does not coincide with the class of k-safe infinitary series over K, and $\mathcal{P}(AP)$.*

Example 42 *Let $AP = \{a, b\}$, $\varphi = \Box(a \wedge 2) \in 2\text{-}\vee\text{-}t\text{-}RULTL(K_2, AP)$. For every $w \in (\mathcal{P}(AP))^{\omega}$, it holds $(\|\varphi\|, w) = 2$ whenever $a \in w(i)$ for*

every $i \geq 0$, *and* $(\|\varphi\|, w) = -\infty$ *otherwise. We verify that* $\|\varphi\|$ *is 2-safe. More precisely, let* $w \in (\mathcal{P}(AP))^{\omega}$ *such that for every* $i > 0$ *there exists* $u \in (\mathcal{P}(AP))^{\omega}$ *with* $(\|\varphi\|, w_{<i}u) \geq 2$. *This implies that for every* $i > 0$, $a \in w(j)$ *for every* $0 \leq j \leq i-1$, *which in turn implies that* $a \in w(i)$ *for every* $i \geq 0$, *i.e.,* $(\|\varphi\|, w) = 2$.

Example 43 *Let* $\varphi = (3 \wedge b)\,\widetilde{U}\,(3 \wedge a) \in 2\text{-}\vee\text{-}t\text{-}RULTL(K_2, AP)$ *where* $AP = \{a, b\}$. *It holds* $(\|\varphi\|, w) = 3$ *if* $b \in w(j)$ *for all* $j \geq 0$, *or if there exists an* $i \geq 0$, *such that* $a \in w(i)$ *and* $b \in w(j)$ *for every* $0 \leq j \leq i-1$. *Otherwise,* $(\|\varphi\|, w) = -\infty$. *Clearly,* $\|\varphi\|$ *is 2-safe. More precisely, let* $w \in (\mathcal{P}(AP))^{\omega}$ *such that for every* $i > 0$ *there exists* $u \in (\mathcal{P}(AP))^{\omega}$ *with* $(\|\varphi\|, w_{<i}u) \geq 2$. *Then, for every* $i > 0$, $\exists u \in (\mathcal{P}(AP))^{\omega}$ *such that at least one of the following is true:* $(\|(3 \wedge b)\,U\,(3 \wedge a)\|, w_{<i}u) \geq 2$, *or* $(\|\Box(3 \wedge b)\|, w_{<i}u) \geq 2$.[3] *This implies that at least one of the following is true: There exist infinitely many* $i > 0$ *such that* $\exists u \in (\mathcal{P}(AP))^{\omega}$ *with* $(\|(3 \wedge b)\,U\,(3 \wedge a)\|, w_{<i}u) \geq 2$, *or there exist infinitely many* $i > 0$ *such that* $\exists u \in (\mathcal{P}(AP))^{\omega}$ *with* $(\|\Box(3 \wedge b)\|, w_{<i}u) \geq 2$. *If the first case is true we derive that* $\forall i > 0, \exists u \in (\mathcal{P}(AP))^{\omega}$, *such that* $(\|(3 \wedge b)\,U\,(3 \wedge a)\|, w_{<i}u) \geq 2$.[4] *Hence,* $\forall i > 0, \exists u \in (\mathcal{P}(AP))^{\omega}, j_i \geq 0$ *such that* $\left(\|3 \wedge b\|, (w_{<i}u)_{\geq h}\right) \geq 2\,(0 \leq h \leq j_i - 1)$, *and* $\left(\|3 \wedge a\|, (w_{<i}u)_{\geq j_i}\right) \geq 2$.[5] *Assume that there exist an* $i > 0$, *such that* $\exists u \in (\mathcal{P}(AP))^{\omega}, j_i \in \{0, \ldots, i-1\}$ *such that* $\left(\|3 \wedge b\|, (w_{<i}u)_{\geq h}\right) \geq 2\,(0 \leq h \leq j_i - 1)$ *and* $\left(\|3 \wedge a\|, (w_{<i}u)_{\geq j_i}\right) \geq 2$. *This implies that* $\exists j \geq 0$ *such that* $(\|3 \wedge b\|, w_{\geq h}) \geq 2\,(0 \leq h \leq j-1)$ *and* $(\|3 \wedge a\|, w_{\geq j}) \geq 2$, *i.e.,* $(\|\psi U \xi\|, w) \geq 2$, *which in turn implies that* $\left(\left\|\psi \widetilde{U} \xi\right\|, w\right) \geq 2$ *as desired. Assume now that for all* $i > 0, \nexists j_i < i$ *such* $\left(\|3 \wedge a\|, (w_{<i}u)_{\geq j_i}\right) \geq 2$. *Then, we get that for all* $i > 0$, $\left(\|3 \wedge b\|, (w_{<i}u)_{\geq h}\right) \geq 2\,(0 \leq i \leq j-1)$, *which implies that* $(\|3 \wedge b\|, w_{\geq i}) \geq 2$ *for all* $i \geq 0$, *i.e.,* $(\|\Box(3 \wedge b)\|, w) \geq 2$, *and thus* $\left(\left\|(3 \wedge b)\,\widetilde{U}\,(3 \wedge a)\right\|, w\right) \geq 2$ *as desired.*

If the second case is true we conclude that $\left(\left\|(3 \wedge b)\,\widetilde{U}\,(3 \wedge a)\right\|, w\right) \geq 2$ *with similar arguments.*

[3] This claim is derived by assuming the contrary, and Lemma 8.

[4] This claim is derived with the same arguments used in the proof of Theorem 38 in the inductive step of the disjunction.

[5] This claim is derived by assuming the contrary, and using the definition of Val^{ω}, and Remark 26.

By Remark 33, Theorems 29, 30, and Corollary 39 we conclude the following.

Corollary 44 *(i) Let K be an idempotent ordered GP-ω-valuation monoid, $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$ and $\varphi \in k\text{-}t\text{-}RULTL(K, AP)$. Then, we can effectively construct a wBa over $\mathcal{P}(AP)$ and K recognizing $\|\varphi\|$.*

(ii) Let K be an idempotent ordered TGP-ω-valuation monoid, $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$ and $\varphi \in k\text{-}\vee\text{-}t\text{-}RULTL(K, AP)$. Then, we can effectively construct a wBa over $\mathcal{P}(AP)$ and K recognizing $\|\varphi\|$.

6 A Motivating Example

In this section we present an example on how k-safe weighted LTL-formulas can be used to draw conclusions related to the quantitative behavior of systems. More specifically, motivated by the work in [30], we consider a weighted transition system that models the routes of a robot that moves between states realizing in some of them gathering of data, or control, or uploading of data. The weights on the transitions of the system represent the amount of energy that is produced by the robot's movement in each transition. In fact the weighted transition system that we consider is a variation of the corresponding one in [30]. We determine quantitative properties that could be related to the preservation of the movement of the robot, and for a specific rational number k, we consider a k-safe weighted LTL formula over K_2 and $\mathcal{P}(AP)$ that expresses the aforementioned properties, where AP is the set of propositions used to label the states of the weighted transition system. We then translate the weighted transition system to a weighted Büchi automaton over K_2 and $\mathcal{P}(AP)$, with the following property: each path of the automaton, with weight different from $-\infty$, simulates a run of the transition system, and vice-versa. We finally prove that the behavior of the automaton is equal to the semantics of the formula, verifying in this way that the weighted transition system has the desirable quantitative properties.

We let $T = (Q, q_0, R, AP, L, we)$ where

- $Q = \{q_0, q_1, q_2, q_3\}$,
- $AP = \{gather, upload, control\}$,
- $R = \{(q_0, q_2), (q_2, q_3), (q_3, q_0), (q_2, q_1), (q_1, q_0)\}$,

- $L : Q \to \mathcal{P}(AP)$ is given by $L(q_0) = \emptyset$, $L(q_1) = \{upload\}$, $L(q_2) = \{gather\}$, $L(q_3) = \{control, upload\}$,
- $we : R \to Q_{\geq 0}$ is defined by $we(q_0, q_2) = 8$, $we(q_2, q_3) = 5$, $we(q_2, q_1) = 7$, $we(q_3, q_0) = 3$, and $we(q_1, q_0) = 10$.

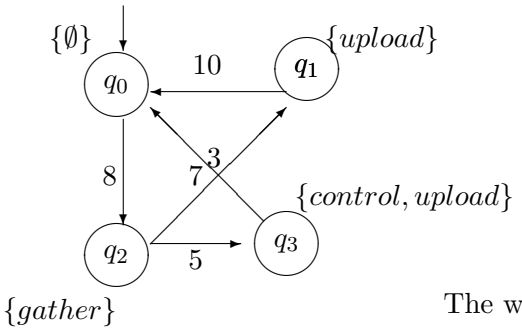

Figure 2
The weighted transition system T

Remark 45 *Let $r_1 = q_0q_1q_2\ldots, r_2 = q'_0q'_1q'_2\ldots$ be runs of T such that $r_1 \neq r_2$, then $\exists i \geq 0$ with $q_i \neq q'_i$, which by the definition of T implies that $\exists i \geq 0$ such that $L(q_i) \neq L(q'_i)$, i.e., $w_{r_1} \neq w_{r_2}$.*

We are interested in verifying that T has the following properties that could be related with the preservation of the movement of the robot.

Property 1*: The greatest amount of energy that is produced infinitely often along any run of T is greater than* 8.

Property 2*: Along any run of T, whenever a state q, where uploading and control are both executed, is visited, then an amount of energy equal to* 8 *should be produced during a transition that appears in the run after a finite number of transitions after q is visited, and whenever a state q', where uploading is executed but control is not, is visited, then an amount of energy equal to* 10 *should be produced during a transition that appears in the run after a finite number of transitions after q' is visited, and whenever a state q'', where uploading is not executed, is visited, then an amount of energy equal to* 8 *should be produced during a transition that appears in the run after a finite number of transitions after q'' is visited.*

To this end we work as follows. We translate T to a wBa over K_2 and AP, such that the sequence of weights of the transitions of each path of the automaton, with weight different from $-\infty$, over a word w simulates the sequence of weights of a run of T defining the word w, and vice-versa.

Moreover, we consider an 8-$\vee$-totally restricted U-nesting weighted LTL-formula φ whose semantics expresses **Properties 1, and 2.** We finally prove that the behavior of φ coincides with the behavior of the automaton.

More specifically, we consider the normalized wBa $\mathcal{M} = (Q, wt, q_0, Q)$ over K_2 and $\mathcal{P}(AP)$ where $wt : Q \times \mathcal{P}(AP) \times Q$ is given by

$$wt(q_i, \pi, q_j) = \begin{cases} we(q_i, q_j) & \text{if } L(q_i) = \pi, \text{ and } (q_i, q_j) \in R \\ -\infty & \text{otherwise} \end{cases}$$

for every $i, j \in \{0, 1, 2, 3\}$. Clearly, every path of $\mathcal{M}$ over a word $w \in (\mathcal{P}(AP))^{\omega}$ is successful.

Next, let $r = q_0 q_1 q_2 \dots$ be a run of T defining the word w_r. Then $P^r_{w_r} = (q_i, L(q_i), q_{i+1})_{i \geqslant 0}$ is a path of $\mathcal{M}$ over w_r with

$$weight_{\mathcal{M}}\left(P^r_{w_r}\right) = \text{limsup}\left(\left(wt(q_i, L(q_i), q_{i+1})\right)_{i \geq 0}\right)$$

$$= \text{limsup}\left(\left(we(q_i, q_{i+1})\right)_{i \geq 0}\right) \neq -\infty.$$

Moreover, if $P_w = (q_i, \pi_i, q_{i+1})_{i \geqslant 0}$ is a path of $\mathcal{M}$ over $w = \pi_0 \pi_1 \pi_2 \dots$ with $weight_{\mathcal{M}}(P_w) \neq -\infty$, then $r = q_0 q_1 q_2 \dots$ is a run of T defining the word $w_r = w$, and the weight sequence $\omega_r = (wt(q_i, \pi_i, q_{i+1}))_{i \geq 0} = (we(q_i, q_{i+1}))_{i \geq 0}$. Thus, there is a one-to-one correspondence between the runs of T defining the word w, and paths of $\mathcal{M}$ over w with weight$\neq -\infty$. Then, by Remark 45, we conclude that there exists at most one path P_w of $\mathcal{M}$ over w with $weight_{\mathcal{M}}(P_w) \neq -\infty$, and $weight_{\mathcal{M}}(P_w)$ represents the maximum amount of energy that is produced infinitely often by the movement of the robot along the corresponding run r of T. Thus,

$$(\|\mathcal{M}\|, w) = \begin{cases} weight_{\mathcal{M}}(P^r_w) & \text{if there exists a run } r \text{ of } T \text{ with } w_r = w \\ -\infty & \text{otherwise} \end{cases}$$

We consider now the formula $\varphi = \psi \wedge \psi' \in 8\text{-}\vee\text{-}t\text{-}RULTL(K_2, AP)$ where

$$\psi' = \Box((\neg upload \wedge 8) \vee (upload \wedge control \wedge 8) \vee (upload \wedge \neg control \wedge 10)),$$

and

$$\psi = \varphi_1 \wedge \Box(\varphi_1 \vee \varphi_2 \vee \varphi_3)$$

where

- $\varphi_1 = \neg gather \wedge \neg upload \wedge \neg control \wedge \bigcirc gather$, and
- $\varphi_2 = gather \wedge \neg upload \wedge \neg control \wedge \bigcirc upload$, and
- $\varphi_3 = ((control \wedge upload \wedge \neg gather) \vee (\neg control \wedge upload \wedge \neg gather)) \wedge \bigcirc (\neg gather \wedge \neg upload \wedge \neg control)$.

We note that φ_1 expresses the property that the robot is at a state where no action is realized, and at the next state that is visited gathering of data should be realized. φ_2 describes the property that the robot is at a state where gathering of data is realized, and no other action is executed by the robot while it is in that state, and at the next state that is visited, uploading of data should be realized. φ_3 expresses the property the robot is at a state, where uploading of data and control are realized, but gathering of data is not, or at a state where only uploading of data is realized, and at the next step a state is visited where no action is realized. In fact ψ specifies the requirements that an infinite word $w \in (\mathcal{P}(AP))^\omega$ should satisfy so that it is obtained by a run of T. More specifically, it holds that if $w \in (\mathcal{P}(AP))^\omega$ is a word defined by a run in T, then $(\|\psi\|, w) = \mathbf{1}$, and vice-versa. Moreover, $(\|\psi\|, w) = \mathbf{1}$, implies $w \models \psi'_b$. Finally, φ expresses **Property 2**, and since φ is 8-safe we get that φ expresses **Property 1** as well.

Then, for every $w = \pi_0\pi_1\pi_2 \ldots \in (\mathcal{P}(AP))^\omega$ we get

$$(\|\varphi\|, w) = \begin{cases} 8 & \text{if there exists a run } r \text{ of } T \text{ such that } w_r = w, \\ & \text{and } q_1 \text{ does not appear infinitely often along } r \\ 10 & \text{if there exists a run } r \text{ of } T \text{ such that } w_r = w, \\ & \text{and } q_1 \text{ appears infinitely often along } r \\ -\infty & \text{otherwise} \end{cases}.$$

It holds

$$\| \varphi \| = \|\mathcal{M}\|.$$

Then, for every $w \in supp(\|\mathcal{M}\|)$, it holds

$$(\| \mathcal{M} \|, w) = \sup_{P_w \in succ_{\mathcal{M}}(w)} (weight_{\mathcal{M}}(P_w)) = weight_{\mathcal{M}}(P_w^r) \geq 8,$$

where r is the unique run of T with $w_r = w$. We thus get that the maximum amount of energy that is produced infinitely often along any run of T is greater than 8, and **Property 2** is satisfied for any run of T.

7 Algorithms for k-safety

In this section, firstly for K_2, K_3 (resp. for K_1), we present an algorithm that given a wBa and a k-safe $\vee$-totally generalized U-nesting weighted LTL formula (resp. k-safe totally generalized U-nesting weighted LTL formula), decides if the semantics of the formula coincides with the behavior of the automaton. The core of the algorithm is based on the fact that the weighted LTL formula can be effectively translated to a wBa, and the quantitative language equivalence problem is decidable for wBa over these specific structures. For the proof of this last decidability result we follow the approach of the constructive proof of the corresponding result in [7]. More precisely, for the proof of the following lemmas we follow the constructive approach of Lemmas 1, and 2 in [7]. We finally conclude that the algorithms presented can be adopted for larger families of TGP-ω-valuation monoids, and GP-ω-valuation monoids.

Lemma 46 *Let $\mathcal{M} = (Q, wt, q_0, F)$ be a wBa over K_3 and A. We can effectively construct a wBa $\widehat{\mathcal{M}} = \left(\widehat{Q}, \widehat{wt}, \widehat{q_0}, \widehat{F}\right)$ over K_2 and A such that $\|\mathcal{M}\| = \left\|\widehat{\mathcal{M}}\right\|$.*

Proof: Let $C = Im\,(wt) \setminus \{\infty\}$. We define the wBa $\widehat{\mathcal{M}} = \left(\widehat{Q}, \widehat{wt}, \widehat{q_0}, \widehat{F}\right)$ over K_2 as follows . We set

- $\widehat{Q} = Q \cup (Q \times C)$
- $\widehat{q_0} = q_0$
- $\widehat{wt}\,(p, a, \widetilde{p}) = \begin{cases} wt\,(p, a, \widetilde{p}) & \text{if } (p, a, \widetilde{p}) \in Q \times A \times Q \\ wt\,(p, a, \widetilde{q}) & \text{if } p \in Q, \widetilde{p} = (\widetilde{q}, wt\,(p, a, \widetilde{q})) \in Q \times C \\ v & \text{if } p = (q, v), \widetilde{p} = (\widetilde{q}, v) \in Q \times C, q \neq \widetilde{q}, \\ & \text{and } wt\,(q, a, \widetilde{q}) \leq v \\ v & \text{if } p = \widetilde{p} = (q, v) \in Q \times C \\ -\infty & \text{otherwise} \end{cases}$
- $\widehat{F} = F \cup (F \times C)$

Let $w = a_0 a_1 \ldots \in A^\omega$. We show that for every $\widehat{P}_w \in succ_{\widehat{\mathcal{M}}}(w)$ with $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) \neq -\infty$, there exists a successful path P_w of $\mathcal{M}$ over w with $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) \leq weight_{\mathcal{M}}(P_w)$. We let $\widehat{P}_w = t_0 t_1 t_2 \ldots \in succ_{\widehat{\mathcal{M}}}(w)$ with $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) \neq -\infty$. Let $In^{\widehat{Q}}\left(\widehat{P}_w\right) \subseteq Q$. Then, $\widehat{P}_w$ is also a successful path of $\mathcal{M}$ over w, and we point out the following cases. (a) If $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) \neq \infty$, then $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) = limsup\left(\widehat{wt}(t_i)\right)_{i\geq 0} \leq weight_{\mathcal{M}}(P_w) = \sup_{-\infty}(wt(t_i))_{i\geq 0}$ as desired. (b) If $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) = \infty$, then the weight of each transition is equal to ∞, and thus $\widehat{P}_w$ is also a successful path of $\mathcal{M}$ with the same weight. Assume now that there exists $q \in In^{\widehat{Q}}\left(\widehat{P}_w\right)$, such that $q \notin Q$. Then, $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) \neq \infty$, and $\widehat{P}_w = (q_0, a_0, q_1)\ldots(q_j, a_j, q_{j+1})\,(q_{j+1}, a_{i+1,}, (q_{i+2}, v))\,((q_{i+j}, v), a_{i+j}, (q_{i+j+1}, v))_{j\geq 2}$ with $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) = v = \widehat{wt}(q_{j+1}, a_{j+1}, (q_{j+2}, v)) = wt(q_{j+1}, a_{j+1}, q_{j+2})$. Then, the path $P_w = (q_i, a_i, q_{i+1})_{i\geq 0}$ is a successful path of $\mathcal{M}$ over w with $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) \leq weight_{\mathcal{M}}(P_w)$ as desired.

Thus,

$$\begin{aligned}\left(\left\|\widehat{\mathcal{M}}\right\|, w\right) &= \sup_{\widehat{P}_w \in succ_{\widehat{\mathcal{M}}}(w)} \left(weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right)\right) \\ &\leq \sup_{P_w \in succ_{\mathcal{M}}(w)} (weight_{\mathcal{M}}(P_w)) = (\|\mathcal{M}\|, w).\end{aligned} \tag{6}$$

We now prove that for every $P_w \in succ_{\mathcal{M}}(w)$ with $weight_{\mathcal{M}}(P_w) \neq -\infty$, there exists a $\widehat{P}_w \in succ_{\widehat{\mathcal{M}}}(w)$ such that $weight_{\mathcal{M}}(P_w) \leq weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right)$. We let $P_w = (t_i)_{i\geq 0} \in succ_{\mathcal{M}}(w)$ with $weight_{\mathcal{M}}(P_w) \neq -\infty$. If $weight_{\mathcal{M}}(P_w) \neq \infty$, then we obtain a successful path $\widehat{P}_w$ of $\widehat{\mathcal{M}}$ over w with $weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right) = weight_{\mathcal{M}}(P_w)$ in the following way: We let $t_i = (q_i, a_i, q_{i+1})$ for all $i \geq 0$, and we construct $\widehat{P}_w$ by copying the finite prefix of P_w until we reach a transition t_i with $wt(t_i) = v = weight_{\mathcal{M}}(P_w)$, and then we act as follows: if $q_{i+1} \in F$ we loop through (q_{i+1}, v), i.e.,

$$\widehat{P}_w = (q_0, a_0, q_1)\ldots(q_i, a_i, (q_{i+1}, v))\,((q_{i+1}, v), a_{i+j}, (q_{i+1}, v))_{j\geq 1}$$

otherwise we reach a final state $(q_{i+k}, v), k \geq 2, q_{i+k} \in F$, with the sequence of transitions

$$(q_{i+1}, a_{i+1}, (q_{i+2}, v))\left((q_{i+2}, v), a_{i+2}, (q_{i+3}, v)\right)\ldots\left((q_{i+k-1}, v), a_{i+k-1}, (q_{i+k}, v)\right)$$

and finally loop through (q_{i+k}, v), i.e.,

$$\begin{aligned}\widehat{P}_w \;=\; & (q_0, a_0, q_1)\ldots(q_i, a_i, (q_{i+1}, v))\,(q_{i+1}, a_{i+1}, (q_{i+2}, v))\\ & \left((q_{i+2}, v), a_{i+2}, (q_{i+3}, v)\right)\ldots\left((q_{i+k-1}, v), a_{i+k-1}, (q_{i+k}, v)\right)\\ & \left((q_{i+k}, v), a_{i+j}, (q_{i+k}, v)\right)_{j\geq k}\end{aligned}$$

If $weight_{\mathcal{M}}(P_w) = \infty$, then P_w is also a successful path of $\widehat{\mathcal{M}}$ over w with the same weight. Thus,

$$\begin{aligned}\left(\|\mathcal{M}\|, w\right) \;&=\; \sup_{P_w \in succ_{\mathcal{M}}(w)} \left(weight_{\mathcal{M}}(P_w)\right) \qquad (7)\\ &\leq\; \sup_{\widehat{P}_w \in succ_{\widehat{\mathcal{M}}}(w)} \left(weight_{\widehat{\mathcal{M}}}\left(\widehat{P}_w\right)\right) = \left(\left\|\widehat{\mathcal{M}}\right\|, w\right).\end{aligned}$$

Hence, by (6), (7) we get that $\left(\left\|\widehat{\mathcal{M}}\right\|, w\right) = \left(\|\mathcal{M}\|, w\right)$ for every $w \in A^\omega$, and this concludes our proof. □

Lemma 47 *Let $\mathcal{N} = (Q, wt, q_0, F)$ be a normalized wBa over K_2 and A, and $v \in \overline{\mathbb{Q}}$. We can effectively construct a Ba $\mathcal{N}^{\geq v}$ such that $\left\|\mathcal{N}^{\geq v}\right\| = \{w \in A^\omega \mid \left(\|\mathcal{N}\|, w\right) \geq v\}$.*

Proof: We first construct a Ma $\mathcal{M}^{\geq v}$ with $\left\|\mathcal{M}^{\geq v}\right\| = \{w \in A^\omega \mid \left(\|\mathcal{N}\|, w\right) \geq v\}$. Then, we obtain the claim by Theorem 1. We assume first that $v \in Q$. We let $\mathcal{M}^{\geq v} = \left(Q^{\geq v}, A, q_0^{\geq v}, \Delta^{\geq v}, F^{\geq v}\right)$ where

- $Q^{\geq v} = (Q \times \{0,1,3,4\} \times \{B, C\}) \cup \left(\widehat{Q} \times \{2,5,6\} \times \{B, C\}\right)$
  where $\widehat{Q} = \{s_q \mid q \in Q\}$ is a set of copies of the elements of Q
- $q_0^{\geq v} = (q_0, 0, C)$
- The set of final sets is defined as follows
  $F^{\geq v} = \left\{\widehat{F} \mid \widehat{F} \subseteq F_1 \cup F_2 \cup F_3, F_1 \cap \widehat{F} \neq \emptyset, F_2 \cap \widehat{F} \neq \emptyset\right\}$ where
  $F_1 = (Q \times \{3\} \times \{C, B\}) \cup \left(\widehat{Q} \times \{2\} \times \{C, B\}\right) \cup \left(\widehat{Q} \times \{6\} \times \{C, B\}\right)$,
  $F_2 = (Q \times \{0,1,3,4\} \times \{B\}) \cup \left(\widehat{Q} \times \{2,5,6\} \times \{B\}\right)$, and
  $F_3 = Q \times \{1,4\} \times \{B, C\}$.

The definition of $\Delta^{\geq v}$, and the proof of the equality

$$\left\|\mathcal{M}^{\geq v}\right\| = \{w \in A^{\omega} \mid (\|\mathcal{N}\|, w) \geq v\}$$

is presented in an analytical way in the Appendix. We only here describe intuitively the operation of $\mathcal{M}^{\geq v}$. The set of states of $\mathcal{M}^{\geq v}$ are triples in which the first component is either a state of $\mathcal{N}$ or a copy of a state of $\mathcal{N}$. A transition $((q,i,k), a, (\overline{q},j,l))$ of $\mathcal{M}^{\geq v}$ between states whose first component is an element of Q simulates the transition $(q,a,\overline{q})$ of $\mathcal{N}$. At the same time the information on whether the weight of the simulated transition is greater or equal to v, or smaller to v, or equal to ∞ is indicated by the number that appears in second component of the arriving state, i.e., j is equal to $1, 3$, or 4 respectively. The third component of the arriving state is B if the arriving state of the original transition is final, and C otherwise. Copies of states of $\mathcal{N}$ are used to simulate paths of $\mathcal{N}$ where there exists a position after which the weight of each transition, appearing in the path, is ∞. More precisely, a path of the original automaton where the weight of each transition is equal to ∞ is simulated by a path of $\mathcal{M}^{\geq v}$ where the first component of every state, but the first one, is a copy of the corresponding state of the original path. The second component of every state, but the first one, of the path of $\mathcal{M}^{\geq v}$ is equal to 2, indicating that the weight of the transition, that is being simulated, is equal to ∞. The third components of the states that appear in the new path are either B, or C, depending on whether the first component of the state is a copy of a final state of the original automaton or not. Finally, paths of the original automaton (with weight different from $\neq -\infty$) in which there are finitely many transitions with a rational weight, and at least one transition with weight greater, or equal to v, are simulated in the new automaton by paths where the automaton transits to a state where the first component is a copy of the corresponding state of $\mathcal{N}$ when the weight of the transition that is being simulated is a rational number greater, or equal to v. The second component of this state equal to 5. Then, the rest of the original original path is simulated by transitions between states whose first component is a copy of an element of Q, and second component is equal to 5, up until the point where some transition with weight ∞ is simulated. Then, $\mathcal{M}^{\geq v}$ enters a state whose second component is equal to 6. Finally, the new automaton can move from a state with second component 6, only to states whose second component is also 6, and only to simulate a transition of the original automaton whose weight is equal to ∞. The Muller acceptance condition is used to verify that successful paths of

the new automaton simulate successful paths of $\mathcal{N}$ in which the weights of the transitions are equal to the desired values infinitely often.

If $v = \infty$, we prove our claim similarly. Finally, if $v = -\infty$, then $(\|\mathcal{N}\|, w) \geq v$ for every $w \in A^\omega$, which implies that $\{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\} = A^\omega$, and it is trivial to construct the desired Ba. □

Lemma 48 *Let $\mathcal{N} = (Q, wt, q_0, F)$ be a normalized wBa over K_1 and A, and $v \in \mathbb{Q}$. We can effectively construct a Ba $\mathcal{N}^{\geq v}$ over A such that $\left\|\mathcal{N}^{\geq v}\right\| = \{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\}$.*

Proof: To prove the Lemma's claim we first construct a Ra $\mathcal{M}^{\geq v}$ with the property that $\left\|\mathcal{M}^{\geq v}\right\| = \{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\}$. Then, the claim is obtained by Theorem 1. The construction of $\mathcal{M}^{\geq v}$ is presented in the Appendix. □

Theorem 49 *The quantitative language-inclusion problem is decidable for wBa over K_1, or K_2, or K_3 and A.*

Proof: Let $\mathcal{N} = (Q, wt, q_0, F)$, $\widehat{\mathcal{N}} = \left(\widehat{Q}, \widehat{wt}, \widehat{q_0}, \widehat{F}\right)$ be two wBa over K_2 and A. Let $C = Im(wt)$, and $\widehat{C} = Im\left(\widehat{wt}\right)$. Clearly, $C, \widehat{C}$ are computable. Since $(\|\mathcal{N}\|, w) \in C$ for all $w \in A^\omega$, it holds $\|\mathcal{N}\| \sqsubseteq \left\|\widehat{\mathcal{N}}\right\|$ iff $\left\|\mathcal{N}^{\geq v}\right\| \subseteq \left\|\widehat{\mathcal{N}}^{\geq v}\right\|$ for all $v \in C$, where $\mathcal{N}^{\geq v}, \widehat{\mathcal{N}}^{\geq v}$ are the Ba obtained by $\mathcal{N}, \widehat{\mathcal{N}}$ following the proof of Lemma 47. We thus conclude that the quantitative inclusion problem is decidable for wBa over K_2. The proof for wBa over K_1 is treated similarly. We conclude our claim for wBa over K_3 by Lemma 46. □

Theorem 50 *The quantitative language equivalence problem is decidable for wBa over K_1, or K_2, or K_3 and A.*

Let now AP be a finite set of atomic propositions. The following Theorem is concluded by Corollary 44, and Theorem 50.

Theorem 51 *(i) Let $k \in K_1 \setminus \{-\infty, \infty\}$. If $\varphi \in k\text{-}t\text{-}RULTL(K_1, AP)$, and $\mathcal{M}$ is a wBa over K_1 and $\mathcal{P}(AP)$, it is decidable whether $\|\varphi\| = \|\mathcal{M}\|$.*

(ii) Let $k \in K_2 \setminus \{-\infty, \infty\}$. If $\varphi \in k\text{-}\vee\text{-}t\text{-}RULTL(K_2, AP)$, and $\mathcal{M}$ is a wBa over K_2 and $\mathcal{P}(AP)$, it is decidable whether $\|\varphi\| = \|\mathcal{M}\|$.

(iii) *Let* $k \in K_3 \setminus \{-\infty, \infty\}$. *If* $\varphi \in k\text{-}\vee\text{-}t\text{-}RULTL(K_3, AP)$, *and* $\mathcal{M}$ *is a wBa over* K_3 *and* $\mathcal{P}(AP)$, *it is decidable whether* $\|\varphi\| = \|\mathcal{M}\|$.

Let now K be a totally ordered set with a minimum, and a maximum element, $\mathbf{0}, \mathbf{1}$ respectively, and M_1, M_2, M_3 be the TGP-ω-valuation monoids as defined in Remark 12. Following the arguments of the proofs of Lemmas 46, 47, 48, and Theorems 49, 50 we obtain corresponding results for M_1, M_2, M_3. By these and Corollary 44 we obtain the following corollary.

Corollary 52 *(i)* *Let* $k \in M_1 \setminus \{\mathbf{0}, \mathbf{1}\}$. *If* $\varphi \in k\text{-}t\text{-}RULTL(M_1, AP)$, *and* $\mathcal{M}$ *is a wBa over* M_1 *and* $\mathcal{P}(AP)$, *it is decidable whether* $\|\varphi\| = \|\mathcal{M}\|$.

(ii) *Let* $k \in M_2 \setminus \{\mathbf{0}, \mathbf{1}\}$. *If* $\varphi \in k\text{-}\vee\text{-}t\text{-}RULTL(M_2, AP)$, *and* $\mathcal{M}$ *is a wBa over* M_2 *and* $\mathcal{P}(AP)$, *it is decidable whether* $\|\varphi\| = \|\mathcal{M}\|$.

(iii) *Let* $k \in M_3 \setminus \{\mathbf{0}, \mathbf{1}\}$. *If* $\varphi \in k\text{-}\vee\text{-}t\text{-}RULTL(M_3, AP)$, *and* $\mathcal{M}$ *is a wBa over* M_3 *and* $\mathcal{P}(AP)$, *it is decidable whether* $\|\varphi\| = \|\mathcal{M}\|$.

We are ready now to present the following algorithms. These can be adopted in a straightforward way for M_2, M_3, M_1 respectively.

Algorithm 1: Decision procedure for K_2

Input : wBa $\mathcal{M} = (Q, wt_{\mathcal{M}}, q_0, F)$ and k-safe $\vee$-totally restricted U-nesting weighted LTL formula φ
Output : "yes" if $\|\mathcal{M}\| = \|\varphi\|$, otherwise "no"
Determine $\widetilde{\varphi} \in \vee\text{-}t\text{-}RULTL(K_2, AP)$ with $\|\varphi\| = \|\widetilde{\varphi}\|$
Construct a wBa $\mathcal{A}_{\widetilde{\varphi}} = \left(S, wt_{\mathcal{A}_{\widetilde{\varphi}}}, p_0, P\right)$ such that $\left\|\mathcal{A}_{\widetilde{\varphi}}\right\| = \|\varphi\|$
Compute $Im\left(wt_{\mathcal{A}_{\widetilde{\varphi}}}\right)$
Compute $Im\left(wt_{\mathcal{M}}\right)$
For every $v \in Im\left(wt_{\mathcal{A}_{\widetilde{\varphi}}}\right)$
  construct the Ba $\mathcal{A}_{\widetilde{\varphi}}^{\geq v}$
  construct the Ba $\mathcal{M}^{\geq v}$
  If $\left\|\mathcal{A}_{\widetilde{\varphi}}^{\geq v}\right\| \not\subseteq \left\|\mathcal{M}^{\geq v}\right\|$
    return "no"
end for

For every $v \in Im\,(wt_{\mathcal{M}})$
construct the Ba $\mathcal{A}_{\widetilde{\varphi}}^{\geq v}$
construct the Ba $\mathcal{M}^{\geq v}$
If $\left\|\mathcal{M}^{\geq v}\right\| \not\subseteq \left\|\mathcal{A}_{\widetilde{\varphi}}^{\geq v}\right\|$
return "no"
end for
return "yes"

Algorithm 2: Decision procedure for K_3

Input : wBa $\mathcal{M} = (Q, wt_{\mathcal{M}}, q_0, F)$ and k-safe $\vee$-totally restricted U-nesting weighted LTL-formula φ
Output : "yes" if $\|\mathcal{M}\| = \|\varphi\|$, otherwise "no"
Determine $\widetilde{\varphi} \in \vee\text{-}t\text{-}RULTL\,(K_3, AP)$ with $\|\varphi\| = \|\widetilde{\varphi}\|$
Construct a wBa $\mathcal{A}_{\widetilde{\varphi}} = \left(S, wt_{\mathcal{A}_{\widetilde{\varphi}}}, p_0, P\right)$ over K_3 such that $\left\|\mathcal{A}_{\widetilde{\varphi}}\right\| = \|\varphi\|$
Construct the wBa $\widehat{\mathcal{A}_{\widetilde{\varphi}}}$, $\widehat{\mathcal{M}}$ over K_2 with $\left\|\widehat{\mathcal{A}_{\widetilde{\varphi}}}\right\| = \left\|\mathcal{A}_{\widetilde{\varphi}}\right\|, \left\|\widehat{\mathcal{M}}\right\| = \|\mathcal{M}\|$
Compute $Im\left(wt_{\widehat{\mathcal{A}_{\widetilde{\varphi}}}}\right)$
Compute $Im\left(wt_{\widehat{\mathcal{M}}}\right)$
For every $v \in Im\left(wt_{\widehat{\mathcal{A}_{\widetilde{\varphi}}}}\right)$
construct the Ba $\widehat{\mathcal{A}_{\widetilde{\varphi}}}^{\geq v}$
construct the Ba $\widehat{\mathcal{M}}^{\geq v}$
If $\left\|\widehat{\mathcal{A}_{\widetilde{\varphi}}}^{\geq v}\right\| \not\subseteq \left\|\widehat{\mathcal{M}}^{\geq v}\right\|$
return "no"
end for
For every $v \in Im\left(wt_{\widehat{\mathcal{M}}}\right)$
construct the Ba $\widehat{\mathcal{A}_{\widetilde{\varphi}}}^{\geq v}$
construct the Ba $\widehat{\mathcal{M}}^{\geq v}$
If $\left\|\widehat{\mathcal{M}}^{\geq v}\right\| \not\subseteq \left\|\widehat{\mathcal{A}_{\widetilde{\varphi}}}^{\geq v}\right\|$
return "no"
end for
return "yes"

Algorithm 3: Decision procedure for K_1

Input : wBa $\mathcal{M} = (Q, wt_{\mathcal{M}}, q_0, F)$ and k-safe totally restricted U-nesting weighted LTL-formula φ
Output : "yes" if $\|\mathcal{M}\| = \|\varphi\|$, otherwise "no"
Determine $\widetilde{\varphi} \in t\text{-}RULTL(K_1, AP)$ with $\|\varphi\| = \|\widetilde{\varphi}\|$
Construct a wBa $\mathcal{A}_{\widetilde{\varphi}} = \left(S, wt_{\mathcal{A}_{\widetilde{\varphi}}}, p_0, P\right)$ such that $\left\|\mathcal{A}_{\widetilde{\varphi}}\right\| = \|\varphi\|$
Compute $Im\left(wt_{\mathcal{A}_{\widetilde{\varphi}}}\right)$
Compute $Im\left(wt_{\mathcal{M}}\right)$
For every $v \in Im\left(wt_{\mathcal{A}_{\widetilde{\varphi}}}\right)$
  construct the Ba $\mathcal{A}_{\widetilde{\varphi}}^{\geq v}$
  construct the Ba $\mathcal{M}^{\geq v}$
  If $\left\|\mathcal{A}_{\widetilde{\varphi}}^{\geq v}\right\| \not\subseteq \left\|\mathcal{M}^{\geq v}\right\|$
    return "no"
end for
For every $v \in Im\left(wt_{\mathcal{M}}\right)$
  construct the Ba $\mathcal{A}_{\widetilde{\varphi}}^{\geq v}$
  construct the Ba $\mathcal{M}^{\geq v}$
  If $\left\|\mathcal{M}^{\geq v}\right\| \not\subseteq \left\|\mathcal{A}_{\widetilde{\varphi}}^{\geq v}\right\|$
    return "no"
end for
return "yes"

8 Conclusion

In this work we have introduced the notion of k-safe infinitary series over idempotent ordered TGP-ω-valuation monoids $(K, +, \cdot, Val^{\omega}, \mathbf{0}, \mathbf{1})$ that satisfy specific properties, where $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$. For every $k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, we define two syntactic fragments of the weighted LTL, the fragment of k-safe totally restricted U-nesting weighted LTL formulas, and the fragment of k-safe $\vee$-totally restricted U-nesting weighted LTL formulas (the later is a subclass of the former) with the property that the semantics of the formulas in these fragments are k-safe infinitary series. Whether a k-safe infinitary series over a TGP-ω-valuation monoid K definable by the weighted LTL over K is also definable by a formula in these fragments remains open. Moreover, for every formula in these fragments, we can effectively construct a wBa over K recognizing its semantics. We present an example of how k-safe weighted LTL formulas can be used to describe quantitative properties

of a weighted transition system, and how we can verify that the system has the desired properties by translating it to a wBa that simulates its runs, and proving that the behavior of the automaton coincides with the semantics of the formula. Finally, for two specific idempotent ordered TGP-ω-valuation monoids (resp. a specific idempotent ordered GP-ω-valuation monoid), we present decision procedures that given a wBa and a k-safe $\vee$-totally restricted U-nesting weighted LTL formula (resp. k-safe totally restricted U-nesting weighted LTL formula), decide if the semantics of the formula coincides with the behavior of the automaton. These results are generalized for specific families of idempotent ordered TGP-ω-valuation monoids, and idempotent ordered GP-ω-valuation monoids.

The study of the complexity of the proposed algorithms, and the study of decidability questions related to the formulas in the proposed fragments is a work in progress. Another possible road is the study of the use of k-safe weighted LTL formulas introduced, in the definition of algorithms that given a weighted transition system, a boolean safety formula, and an ω-valuation function, decide whether the words defined by runs of the weighted transition system satisfy the specification, and at the same time corresponding infinite sequences of weights are valuated through the ω-valuation function to a price above a given threshold. A natural theoretical next step is the definition and study of other kinds of quantitative properties of infinitary series over TGP-ω-valuation monoids, like weighted liveness, in the framework of the weighted LTL introduced in this work. The use of temporal logic and model checking techniques in the description and verification of properties of biological systems has been proposed, and is being studied in the literature [15]. We see the use of the weighted LTL and of k-safe weighted LTL formulas in the description, and verification of quantitative properties of such systems as a challenging perspective.

Acknowledgement The author is grateful to the three anonymous reviewers whose valuable remarks led to significant improvements of the content of this work.

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Appendix

Proof: [Proof of Example 10]

We will prove first that K_2 is an idempotent ordered TGP-ω-valuation monoid that satisfy Properties 2, 3, 4, and 5 . Property 3 follows directly from the definition of K_2. First, we prove Property 1 for TGP-ω-valuation monoids.

We consider $L \subseteq_{fin} K_2$, finite index sets $I_j (j \geq 0)$, and $k_{i_j} \in L\,(i_j \in I_j)$ such that for all $j \geq 0$, it holds $k_{i_j} \in L \backslash \{\infty, -\infty\}$ for all $i_j \in I_j$, or $k_{i_j} \in \{\infty, -\infty\}$ for all $i_j \in I_j$. We will prove that

$$\text{limsup} \left(\sup_{i_j \in I_j} k_{i_j} \right)_{j \in \mathbb{N}} = \sup_{(i_j)_j \in I_0 \times I_1 \times \ldots} \left(\text{limsup} \left(k_{i_j} \right)_{j \in \mathbb{N}} \right).$$

We set $A = \text{limsup} \left(\sup_{i_j \in I_j} k_{i_j} \right)_{j \in \mathbb{N}}$, and $B = \sup_{(i_j)_j \in I_0 \times I_1 \times \ldots} \left(\text{limsup} \left(k_{i_j} \right)_{j \in \mathbb{N}} \right)$.

Assume that there exists an $l \geq 0$, such that $k_{i_l} = -\infty$ for all $i_l \in I_l$. Then, for all $(i_j)_j \in I_0 \times I_1 \times \ldots$, $\text{limsup}\big(k_{i_j}\big)_{j \in \mathbb{N}} = -\infty$, i.e., $B = -\infty$. Moreover, $\sup_{i_l \in I_l} k_{i_l} = -\infty$, and thus $A = -\infty$ as desired.

Otherwise, we point out the following cases:

(I) Assume that for all $j \geq 0$ there exists $i_j \in I_j$ such that $k_{i_j} = \infty$. Then, there exist $(i_j)_j \in I_0 \times I_1 \times \ldots$ such that $\text{limsup}\big(k_{i_j}\big)_{j \in \mathbb{N}} = \infty$ which implies that $B = \infty$. In addition, we get that $\sup_{i_j \in I_j} k_{i_j} = \infty$ for all $j \geq 0$, i.e., $A = \infty$ as well.

(II) Assume that there exists finitely many $j \geq 0$, such that $k_{i_j} \neq \infty$ for all $i_j \in I_j$, then $\sup_{i_j \in I_j} k_{i_j} \neq \infty$ for only a finite number of $j \geq 0$, which implies that

$$A = \sup \left\{ \sup_{i_j \in I_j} k_{i_j} \mid j \geq 0 \text{ with} \sup_{i_j \in I_j} k_{i_j} \neq \infty \right\}.$$

Moreover,

$$B = \sup_{\substack{(i_j)_j \in I_0 \times I_1 \times \dots \\ k_{i_j} \neq -\infty, j \geq 0}} \left(\operatorname{limsup} \left(k_{i_j} \right)_{j \in \mathbb{N}} \right)$$
$$= \sup_{\substack{(i_j)_j \in I_0 \times I_1 \times \dots \\ k_{i_j} \neq -\infty, j \geq 0}} \left(\sup \left\{ k_{i_j} \mid j \geq 0 \text{ with } k_{i_j} \neq \infty \right\} \right)$$
$$= \sup \left\{ \sup_{i_j \in I_j} k_{i_j} \mid j \geq 0 \text{ with} \sup_{i_j \in I_j} k_{i_j} \neq \infty \right\} = A$$

where the second equality holds since for all $j \geq 0$, it holds $k_{i_j} \in L \setminus \{\infty, -\infty\}$ for all $i_j \in I_j$, or $k_{i_j} \in \{\infty, -\infty\}$ for all $i_j \in I_j$.

(III) Assume that there exists infinitely many $j \geq 0$ such that $k_{i_j} \neq \infty$ for all $i_j \in I_j$. Since I_k are finite for every $h \geq 0$, it holds $\sup_{i_h \in I_h} (k_{i_h}) \in \{k_{i_h} \mid i_h \in I_h\}$ for every $h \geq 0$, hence there exist a sequence $(i_j)_j \in I_0 \times I_1 \times I_2 \times \dots$ such that $A = \operatorname{limsup} \left(k_{i_j} \right)_{j \geq 0}$, and thus $A \leq B$.

Let now $(i_j)_j \in I_0 \times I_1 \times I_2 \times \dots$,[6] then

$$\begin{aligned} \operatorname{limsup} \left(k_{i_j} \right)_{j \in \mathbb{N}} &= \inf_{j \geq 0} \left(\sup \left\{ k_{i_l} \mid l \geq j, k_{i_l} \neq \infty \right\} \right) \\ &\leq \inf_{j \geq 0} \left(\sup \left\{ \sup_{i_j \in I_j} k_{i_l} \mid l \geq j, \sup_{i_l \in I_l} k_{i_l} \neq \infty \right\} \right) \\ &= \operatorname{limsup} \left(\sup_{i_j \in I_j} k_{i_j} \right)_{j \in \mathbb{N}} = A \end{aligned}$$

where the inequality holds due to the following: for every $j \geq 0$, it holds $k_{i_j} \in L \setminus \{\infty, -\infty\}$ for all $i_j \in I_j$, or $k_{i_j} \in \{\infty, -\infty\}$ for all $i_j \in I_j$, thus for every $j \geq 0$, $k_{i_j} \neq \infty$, implies $\sup_{i_j \in I_j} k_{i_j} \neq \infty$, and vice-versa. So, $A = B$ as desired.

We prove now Property 2. We will prove that for all $k, k_i \, (i \geq 1) \in \overline{\mathbb{Q}}$,

$$\operatorname{limsup} (\infty, k_1, k_2, \dots) = \operatorname{limsup} (k_1, k_2, \dots) .$$

First we assume that $\exists i \geq 1$, such that $k_i = -\infty$. Then,

$$\operatorname{limsup} (\infty, k_1, k_2, \dots) = \operatorname{limsup} (k_1, k_2, \dots) = -\infty.$$

[6]Without any loss we assume that $k_{i_j} \neq -\infty$ for all $j \geq 0$.

Otherwise we point out the following cases.

- If $\forall i \geq 1, k_i = \infty$, limsup($\infty$,$k_1, k_2, \ldots$) =limsup($k_1, k_2, \ldots$) $= \infty$.
- If there exist infinitely many $i \geq 1$ such that $k_i \neq \infty$, then limsup(∞,$k_1, \ldots$) $= \inf\limits_{j \geq 1} (\sup \{k_{i_l} \mid l \geq j, k_{i_l} \neq \infty\})$ =limsup($k_1, \ldots$).
- Finally, if there exist finitely many $i \geq 1$ such that $k_i \neq \infty$, then limsup(∞,$k_1, k_2, \ldots$) $= \sup \left\{k_{i_j} \mid j \geq 1, k_{i_j} \neq \infty\right\}$ =limsup($k_1, k_2, \ldots$).

We prove now Property 4. By definition it holds $k \leq \infty$ for every $k \in \overline{\mathbb{Q}}$.

We prove now Property 5. Let k, $k_i \in \overline{\mathbb{Q}}$ $(i \geq 0)$ such that for every $i \geq 0, k_i \geq k$. We point out the cases $k = \infty, k = -\infty$ or $k \neq \infty, -\infty$. If $k = \infty$, then for all $i \geq 0$, it holds $k_i = \infty$, which implies that limsup$(k_0, k_1, k_2, \ldots) = \infty \geq k$ as wanted. Assume now that $k = -\infty$, then it trivially holds limsup$(k_0, k_1, k_2, \ldots) \geq -\infty$. Finally, if $k \neq \infty, -\infty$, then $k_i \neq -\infty$ for all $i \geq 0$, and thus limsup$(k_0, k_1, k_2, \ldots) = k_i \geq k$ for some $i \geq 0$.[7]

□

Proof: [Proof of Example 11]

We will prove now that K_3 is an idempotent TGP-ω-valuation monoid that satisfy Properties 2, 3, 4, and 5. Property 3 follows directly from the definition of K_3.

We will first prove Property 1 for TGP-ω-valuation monoids. Let $L \subseteq_{fin} \overline{\mathbb{Q}}$, finite index sets I_j, and $k_{i_j} \in L\,(i_j \in I_j)$ such that for all $j \geq 0$, it holds $k_{i_j} \in L \backslash \{\infty, -\infty\}$ for all $i_j \in I_j$, or $k_{i_j} \in \{\infty, -\infty\}$ for all $i_j \in I_j$. We will prove that

$$\sup_{-\infty} \left(\sup_{i_j \in I_j} k_{i_j} \right) = \sup_{(i_j)_j \in I_0 \times I_1 \times \ldots} \left(\sup_{-\infty} \left(k_{i_j}\right)_{j \in \mathbb{N}} \right).$$

We set $A = \sup_{-\infty} \left(\sup\limits_{i_j \in I_j} k_{i_j} \right)$, and $B = \sup\limits_{(i_j)_j \in I_0 \times I_1 \times \ldots} \left(\sup_{-\infty} \left(k_{i_j}\right)_{j \in \mathbb{N}} \right)$. We will prove that $A = B$.

[7] Observe that limsup is defined over $\left(\overline{\mathbb{Q}}_{fin}\right)^{\omega}$, i.e., for some finite subset B of $\overline{\mathbb{Q}}$, $k_i \in B$, for every $i \geq 0$.

- Assume that there exists a $j \geq 0$, such that $k_{i_j} = -\infty$ for all $i_j \in I_j$. Then, $A = B = -\infty$.

- Assume that for all $j \geq 0$, $\exists k_{i_j} \in I_j$ with $k_{i_j} \neq -\infty$, and for all $j \geq 0$ there exists an $i_j \in I_j$ such that $k_{i_j} = \infty$. Then, using the same arguments as in (I) in the previous proof we get that $A = B = \infty$.

- Finally we assume that for all $j \geq 0$, $\exists k_{i_j} \in I_j$ with $k_{i_j} \neq -\infty$, and there exist at least one $j \geq 0$ with $k_{i_j} \neq \infty$ for all $i_j \in I_j$. Then, $\sup_{i_j}\left(k_{i_j}\right) \neq -\infty$ for all $j \geq 0$, and since there exists $j \geq 0$ with $\sup_{i_j \in I_j}\left(k_{i_j}\right) \neq \infty$ we have

$$A = \sup\left\{ \sup_{i_j \in I_j} k_{i_j} \mid j \geq 0, \sup_{i_j \in I_j} k_{i_j} \neq \infty \right\},$$

and

$$B = \sup_{(i_j)_j \in I_0 \times I_1 \times \ldots} \left(\sup\left\{k_{i_j} \mid j \geq 0, k_{i_j} \neq \infty\right\}\right).$$

With arguments similar with the ones used in case (III) in the previous proof we get that it holds $A = B$.

Property 2 is proved with similar arguments used in the proof of the corresponding property for Example 10. We prove now Property 4. By definition it holds $k \leq \infty$ for every $k \in \overline{\mathbb{Q}}$.

Finally, we prove Property 5. Let k, $k_i \in \overline{\mathbb{Q}}\,(i \geq 0)$ such that for every $i \geq 0$ $k_i \geq k$. We point out the cases $k = \infty, k = -\infty$ or $k \neq \infty, -\infty$. If $k = \infty$, then for all $i \geq 0$, it holds $k_i = \infty$, which implies that $\sup_{-\infty}(k_0, k_1, k_2, \ldots) = \infty \geq k$ as desired. Assume now that $k = -\infty$, then it trivially holds $\sup_{-\infty}(k_0, k_1, k_2, \ldots) \geq -\infty$. Finally, if $k \neq \infty, -\infty$, then $k_i \neq -\infty$ for all $i \geq 0$, and thus $\sup_{-\infty}(k_0, k_1, k_2, \ldots) = k_i \geq k$ for some $i \geq 0$. □

Proof: [Proof of Theorem 30] We follow the inductive proof of [22, Theorem 1], and then conclude our claim by the generalization of [22, Lemma 5] for idempotent ordered TGP-ω-valuation monoids. We only present some clarifications on why the arguments in the inductive proof of [22, Theorem 1] can be adopted.

In the proof of [22, Lemma 8, page 253], we justify the equality

$$\begin{aligned} & Val^{\omega}\left(k_1 + k_2, wt\left(B_{\varphi^1_{re}}, \pi_1, B_{\varphi_2}\right), wt\left(B_{\varphi^2_{re}}, \pi_2, B_{\varphi^2}\right), \ldots\right) \\ = \; & Val^{\omega}\left(k_1, wt\left(B_{\varphi^1_{re}}, \pi_1, B_{\varphi_2}\right), wt\left(B_{\varphi^2_{re}}, \pi_2, B_{\varphi^2}\right), \ldots\right) \\ & + Val^{\omega}\left(k_2, wt\left(B_{\varphi^1_{re}}, \pi_1, B_{\varphi_2}\right), wt\left(B_{\varphi^2_{re}}, \pi_2, B_{\varphi^2}\right), \ldots\right) \end{aligned}$$

as follows: Due to the restriction on the fragment, $k_i \neq \mathbf{0}$ implies $k_i \in K \setminus \{\mathbf{0}, \mathbf{1}\}$ for $i \in \{1, 2\}$, and thus it suffices to point out the following cases:

Case 1 *If* $k_1 = \mathbf{0}$, *and* $k_2 \neq \mathbf{0}$, *then*

$$k_1 + k_2 = k_2 \text{ and } Val^{\omega}\left(k_1, wt\left(B_{\varphi^1_{re}}, \pi_1, B_{\varphi_2}\right), wt\left(B_{\varphi^2_{re}}, \pi_2, B_{\varphi^2}\right), \ldots\right) = \mathbf{0},$$

which implies

$$\begin{aligned} & Val^{\omega}\left(k_1 + k_2, wt\left(B_{\varphi^1_{re}}, \pi_1, B_{\varphi_2}\right), wt\left(B_{\varphi^2_{re}}, \pi_2, B_{\varphi^2}\right), \ldots\right) \\ = \; & Val^{\omega}\left(k_2, wt\left(B_{\varphi^1_{re}}, \pi_1, B_{\varphi_2}\right), wt\left(B_{\varphi^2_{re}}, \pi_2, B_{\varphi^2}\right), \ldots\right) \\ = \; & Val^{\omega}\left(k_1, wt\left(B_{\varphi^1_{re}}, \pi_1, B_{\varphi_2}\right), wt\left(B_{\varphi^2_{re}}, \pi_2, B_{\varphi^2}\right), \ldots\right) \\ & + Val^{\omega}\left(k_2, wt\left(B_{\varphi^1_{re}}, \pi_1, B_{\varphi_2}\right), wt\left(B_{\varphi^2_{re}}, \pi_2, B_{\varphi^2}\right), \ldots\right) \end{aligned}$$

Case 2 *If* $k_1 \neq \mathbf{0}$, *and* $k_2 = \mathbf{0}$, *we prove the equality with the same arguments as in the previous case.*

Case 3 *If* $k_1 = k_2 = \mathbf{0}$, *then both parts of the equality are equal to* $\mathbf{0}$.

Case 4 *If* $k_1, k_2 \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, *the equality is obtained by the distributivity of* Val^{ω} *over finite sums for TGP-*ω*-valuation monoids (Property 1).*

Moreover, in the proof of [22, Lemma 8, page 254], we obtain the inequality

$$weight_{\mathcal{A}_{\varphi}}(P_w) \geq weight_{\mathcal{A}_{\psi}}\left(P^1_w\right)$$

by Lemma 7 (of the current work). The use of Lemma 7 is possible since the following hold: In the case $\psi^1 \in next\left(M_{B'_{\varphi},\xi}\right)$, due to the restrictions on the fragment it holds : $wt_1\left(B'_{\psi}, \pi_0, B_{\psi^1}\right) \neq \mathbf{0}$ implies $wt_1\left(B'_{\psi}, \pi_0, B_{\psi^1}\right) \in K \setminus \{\mathbf{0}, \mathbf{1}\}$, and $v_{M_{B_{\varphi'},\xi}}\left(\xi^1\right) \neq \mathbf{0}$ implies $v_{M_{B_{\varphi'},\xi}}\left(\xi^1\right) \in K \setminus \{\mathbf{0}, \mathbf{1}\}$. Also, since K is totally ordered, it holds $wt_1\left(B'_{\psi}, \pi_0, B_{\psi^1}\right) + v_{M_{B_{\varphi'},\xi}}\left(\xi^1\right) \in \left\{wt_1\left(B'_{\psi}, \pi_0, B_{\psi^1}\right), v_{M_{B_{\varphi'},\xi}}\left(\xi^1\right)\right\}$.

In the proofs of [22, Lemmas 9 and 13], we use Lemma 7 (of the current work), instead of of [22, Lemma 3ii].

From the constructive proofs of [22, Lemma 12] (resp. [22, Lemma 14]), we can further derive that for $\psi \in bLTL(K, AP)$, and $\xi \in r\text{-}stLTL(K, AP)$, or $\xi = \lambda U \zeta$, or $\xi = \Box\zeta$ with $\lambda, \zeta \in r\text{-}stLTL(K, AP)$, it holds $v_{B_{\varphi}}(\varphi') = v_{B_{\psi}}(\psi') \cdot v_{B_{\xi}}(\xi') = \mathbf{1}$, or $(v_{B_{\varphi}}(\varphi') \neq \mathbf{1}$ and $v_{B_{\psi}}(\psi') \cdot v_{B_{\xi}}(\xi') \neq \mathbf{1})$ (resp. for

$\psi, \xi \in r\text{-}stLTL\,(K, AP)$, we can further conclude that $v_{B_\varphi}(\varphi') = v_{B_\psi}(\psi') \cdot \prod_{1\leq j\leq k} v_{B_{\xi_j}}\left(\xi'_j\right) = \mathbf{1}$, or $(v_{B_\varphi}(\varphi') \neq \mathbf{1}$ and $v_{B_\psi}(\psi') \cdot \prod_{1\leq j\leq k} v_{B_{\xi_j}}\left(\xi'_j\right) \neq \mathbf{1})$.

In the proof of [22, Lemma 16, page 271], the last equality is true because of the following: For every $i \geq 0$, it holds

$$\begin{aligned}
&\sum_{\substack{k_j \in pri_{\mathcal{A}_\psi}(w_{\geq j})(0\leq j<i)\\ k_i \in pri_{\mathcal{A}_\xi}(w_{\geq i})}} Val^\omega\,(k_0, \ldots, k_{i-1}, k_i, \mathbf{1}, \mathbf{1}, \mathbf{1}, \ldots)\\
= &\sum_{\substack{k_j \in pri_{\mathcal{A}_\psi}(w_{\geq j})\setminus\{\mathbf{0}\}(0\leq j<i)\\ k_i \in pri_{\mathcal{A}_\xi}(w_{\geq i})\setminus\{\mathbf{0}\}}} Val^\omega\,(k_0, \ldots, k_{i-1}, k_i, \mathbf{1}, \mathbf{1}, \mathbf{1}, \ldots)\\
= &\; Val^\omega\left(\begin{array}{c}\sum\limits_{k_0 \in pri_{\mathcal{A}_\psi}(w_{\geq 0})\setminus\{\mathbf{0}\}} k_0, \ldots, \sum\limits_{k_{i-1} \in pri_{\mathcal{A}_\psi}(w_{\geq i-1})\setminus\{\mathbf{0}\}} k_{i-1},\\ \sum\limits_{k_i \in pri_{\mathcal{A}_\xi}(w_{\geq i})\setminus\{\mathbf{0}\}} k_i, \mathbf{1}, \mathbf{1}, \mathbf{1}, \ldots\end{array}\right)\\
= &\; Val^\omega\left(\begin{array}{c}\sum\limits_{k_0 \in pri_{\mathcal{A}_\psi}(w_{\geq 0})} k_0, \ldots, \sum\limits_{k_{i-1} \in pri_{\mathcal{A}_\psi}(w_{\geq i-1})} k_{i-1},\\ \sum\limits_{k_i \in pri_{\mathcal{A}_\xi}(w_{\geq i})} k_i, \mathbf{1}, \mathbf{1}, \mathbf{1}, \ldots\end{array}\right)
\end{aligned}$$

where the first equality is obtained by the fact that $Val_\omega\left((d_i)_{i\geq 0}\right) = \mathbf{0}$ whenever $d_i = \mathbf{0}$ for some $i \geq 0$, and by the completeness axioms of the monoid. The second equality is now obtained by the distributivity of Val^ω over finite sums for TGP-ω-valuation monoids (Property 1). Observe that since $\psi, \xi \in r\text{-}stLTL\,(K, AP)$ it holds: (i) $k_j \in pri_{\mathcal{A}_\psi}(w_{\geq j}) \setminus \{\mathbf{0}\}$ implies $k_j \in K\setminus\{\mathbf{0}, \mathbf{1}\}$ for every $0 \leq j < i$, and (ii) $k_i \in pri_{\mathcal{A}_\xi}(w_{\geq i}) \setminus \{\mathbf{0}\}$ implies $k_i \in K\setminus\{\mathbf{0}, \mathbf{1}\}$.

In the proof of [22, Lemma 16, page 274], we use Lemma 7 of the current work instead of of [22, Lemma 3]. The use of Lemma 7 is possible because of the following: By the fact that $\psi, \xi \in r\text{-}stLTL(K, AP)$, and since K is totally ordered, we can further conclude that the sets $\left\{wt_1\left(B_{\psi^{(m,0)}_{re}}, \pi_m, B_{\psi^{(m,1)}}\right), wt\left(B_{\varphi^m_{re}}, \pi_m, B_{\varphi^{m+1}}\right)\right\}$ (where $0 \leq m < l$) and $\left\{wt_2\left(B_{\xi^0_{re}}, \pi_l, B_{\xi^1}\right), wt\left(B_{\varphi^l_{re}}, \pi_l, B_{\varphi^{l+1}}\right)\right\}$ are subsets of $K\setminus\{\mathbf{0}, \mathbf{1}\}$, and since φ^k_{re} is boolean for every $k > l$, we also get that $wt\left(B_{\varphi^k_{re}}, \pi_k, B_{\varphi^{k+1}}\right) = \mathbf{1}$. In the proof of [22, Lemma 16, page 275], the second implication is justified

with the same arguments we used for the justification of the equality in [22, page 275].

In the proof of [22, Lemma 17, page 276], we use Lemma 7, instead of [22, Lemma 3ii], since $\psi \in r\text{-}stLTL(K, AP)$ and K is totally ordered, we have that $pri_{\mathcal{A}_\psi}(w_{\geq i})$ is finite for every $i \geq 0$, and $\sum_{k_i \in pri_{\mathcal{A}_\psi}(w_{\geq i})} k_i \neq \mathbf{1}$ with $\sum_{k_i \in pri_{\mathcal{A}_\psi}(w_{\geq i})} k_i = \sum_{k_i \in pri_{\mathcal{A}_\psi}(w_{\geq i}) \setminus \{\mathbf{0}\}} k_i \in pri_{\mathcal{A}_\psi}(w_{\geq i}) \setminus \{\mathbf{0}\}$. We use similar arguments to justify the use of Lemma 7 (of the current work) and of the Distributivity of Val^ω over finite sums for TGP-ω-valuation monoids (Property 1) in the rest of the proof.

For the Definitions of $v_{B_\varphi}(\varphi)$, and $pri_{\mathcal{M}}(w)$, where ($\varphi \in LTL(K, AP)$, $\mathcal{M}$ is an ϵ-wgBa, $w \in (\mathcal{P}(AP))^\omega$), we refer the reader to [22]. □

Proof: [Proof of Lemma 47]

Let $v \in Q$, and $\mathcal{M}^{\geq v} = \left(Q^{\geq v}, A, q_0^{\geq v}, \Delta^{\geq v}, F^{\geq v}\right)$ where

- $Q^{\geq v} = (Q \times \{0,1,3,4\} \times \{B,C\}) \cup \left(\widehat{Q} \times \{2,5,6\} \times \{B,C\}\right)$
  where $\widehat{Q} = \{s_q \mid q \in Q\}$ is a set of copies of the elements of Q
- $q_0^{\geq v} = (q_0, 0, C)$
- For all $((q,i,k), a, (\overline{q}, j, l)) \in Q^{\geq v} \times A \times Q^{\geq v}$, it holds $((q,i,k), a, (\overline{q}, j, l)) \in \Delta^{\geq v}$ iff
  1. $i = 0$ and $j = 1, k = l = C$, and $q = q_0, \overline{q} \in Q \setminus F$ and $wt(q, a, \overline{q}) < v$ with $wt(q, a, \overline{q}) \neq -\infty$, or
  2. $i = 0$ and $j = 1, k = C, l = B$ and $q = q_0, \overline{q} \in F$ and $wt(q, a, \overline{q}) < v$ with $wt(q, a, \overline{q}) \neq -\infty$, or
  3. $i = 0$ and $j = 3, k = l = C$, and $q = q_0, \overline{q} \in Q \setminus F, wt(q, a, \overline{q}) \geq v$, and $wt(q, a, \overline{q}) \neq \infty$, or
  4. $i = 0$ and $j = 3, k = C, l = B$, and $q = q_0, \overline{q} \in F, wt(q, a, \overline{q}) \geq v$, and $wt(q, a, \overline{q}) \neq \infty$, or
  5. $i = 0$ and $j = 4, k = l = C$, and $q = q_0, \overline{q} \in Q \setminus F, wt(q, a, \overline{q}) = \infty$, or
  6. $i = 0$ and $j = 4, k = C, l = B$, and $q = q_0, \overline{q} \in F, wt(q, a, \overline{q}) = \infty$, or

7. $i = 0$ and $j = 2$, $k = l = C$, and $q = q_0, \overline{q} = s_{\widetilde{q}} \in \widehat{Q}, \widetilde{q} \notin F, wt\left(q_0, a, \widetilde{q}\right) = \infty$, or
8. $i = 0$ and $j = 2$, $k = C, l = B$, and $q = q_0, \overline{q} = s_{\widetilde{q}} \in \widehat{Q}, \widetilde{q} \in F, wt\left(q_0, a, \widetilde{q}\right) = \infty$, or
9. $i = 2$ and $j = 2$, $k \in \{C, B\}, l = B$, and $q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \in F$, $wt\left(q'', a, \widetilde{q}\right) = \infty$, or
10. $i = 2$ and $j = 2$, $k \in \{C, B\}, l = C$, and $q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \notin F$, $wt\left(q'', a, \widetilde{q}\right) = \infty$, or
11. $i = 1, 3, 4$ and $j = 3$, $k \in \{C, B\}, l = B, q \in Q, \overline{q} \in F$, and $wt\left(q, a, \overline{q}\right) \geq v$, and $wt\left(q, a, \overline{q}\right) \neq \infty$, or
12. $i = 1, 3, 4$ and $j = 3$, $k \in \{C, B\}, l = C, q \in Q, \overline{q} \in Q \backslash F$, and $wt\left(q, a, \overline{q}\right) \geq v$, and $wt\left(q, a, \overline{q}\right) \neq \infty$, or
13. $i = 1, 3, 4$ and $j = 1$, $k \in \{C, B\}, l = B, q \in Q, \overline{q} \in F$, and $wt\left(q, a, \overline{q}\right) < v, wt\left(q, a, \overline{q}\right) \neq -\infty$, or
14. $i = 1, 3, 4$ and $j = 1$, $k \in \{C, B\}, l = C, q \in Q, \overline{q} \in Q \backslash F$, and $wt\left(q, a, \overline{q}\right) < v, wt\left(q, a, \overline{q}\right) \neq -\infty$, or
15. $i = 1, 3, 4$ and $j = 4$, $k \in \{C, B\}, l = C, q \in Q, \overline{q} \in Q \backslash F$, and $wt\left(q, a, \overline{q}\right) = \infty$, or
16. $i = 1, 3, 4$ and $j = 4$, $k \in \{C, B\}, l = B, q \in Q, \overline{q} \in F$, and $wt\left(q, a, \overline{q}\right) = \infty$, or
17. $i = 3$, and $j = 5, k \in \{C, B\}, l = C$, and $q \in Q, \overline{q} = s_{\widetilde{q}} \in \widehat{Q}, \widetilde{q} \notin F$, $wt\left(q, a, \widetilde{q}\right) \neq -\infty$, or
18. $i = 3$, and $j = 5$, $k \in \{C, B\}, l = B$, and $q \in Q, \overline{q} = s_{\widetilde{q}} \in \widehat{Q}, \widetilde{q} \in F$, $wt\left(q, a, \widetilde{q}\right) \neq -\infty$, or
19. $i = 5$, and $j = 5$, $k \in \{C, B\}, l = C$, and $q, \overline{q} \in \widehat{Q}, q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \notin F, wt\left(q'', a, \widetilde{q}\right) \neq -\infty$, or
20. $i = 5$, and $j = 5$, $k \in \{C, B\}, l = B$, and $q, \overline{q} \in \widehat{Q}, q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \in F, wt\left(q'', a, \widetilde{q}\right) \neq -\infty$, or
21. $i = 5$, and $j = 6$, $k \in \{C, B\}, l = C$, and $q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \notin F, wt\left(q'', a, \widetilde{q}\right) = \infty$, or
22. $i = 6$, and $j = 6$, $k \in \{C, B\}, l = C$, and $q = s_{q''}, \widetilde{q} \notin F, \overline{q} = s_{\widetilde{q}}, wt\left(q'', a, \widetilde{q}\right) = \infty$, or
23. $i = 5$, and $j = 6$, $k \in \{C, B\}, l = B$, and $q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \in F, wt\left(q'', a, \widetilde{q}\right) = \infty$, or

24. $i = 6$, and $j = 6$, $k \in \{C, B\}, l = B$, and $q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \in F, wt(q'', a, \widetilde{q}) = \infty$.

- The set of final sets is defined as follows

$$F^{\geq v} = \left\{ \widehat{F} \mid \widehat{F} \subseteq F_1 \cup F_2 \cup F_3, F_1 \cap \widehat{F} \neq \emptyset, F_2 \cap \widehat{F} \neq \emptyset \right\}$$

$$F_1 = (Q \times \{3\} \times \{C, B\}) \cup \left(\widehat{Q} \times \{2\} \times \{C, B\}\right) \cup \left(\widehat{Q} \times \{6\} \times \{C, B\}\right)$$

$$F_2 = (Q \times \{0, 1, 3, 4\} \times \{B\}) \cup \left(\widehat{Q} \times \{2, 5, 6\} \times \{B\}\right)$$

$$F_3 = Q \times \{1, 4\} \times \{B, C\}$$

We prove now that $\left\|\mathcal{M}^{\geq v}\right\| = \{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\}$. We let first $w = a_0 a_1 a_2 \ldots \in \{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\}$. This implies that

$$(\|\mathcal{N}\|, w) = \sup_{P_w \in succ_{\mathcal{N}}(w)} (weight_{\mathcal{N}}(P_w)) = \infty$$

or

$$(\|\mathcal{N}\|, w) = \sup_{P_w \in succ_{\mathcal{N}}(w)} (weight_{\mathcal{N}}(P_w)) = v'' \in \mathbb{Q} \text{ with } v'' \geq v.$$

If the first case is true there exists a successful path $P_w = (q_i, a_i, q_{i+1})_{i \geq 0}$ of $\mathcal{N}$ over w such that $wt(q_i, a_i, q_{i+1}) = \infty$ for all $i \geq 0$. We consider the path

$$P''_w = ((q_0, 0, C), a_0, (s_{q_1}, 2, D_1)) \left((s_{q_i}, 2, D_i), a_i, \left(s_{q_{i+1}}, 2, D_{i+1}\right)\right)_{i \geq 1}$$

where $D_i = B$ if $q_i \in F$, and $D_i = C$ if $q_i \notin F$ for every $i \geq 1$. (We shall call such a path P''_w a path of form A). Clearly, P''_w is a successful path of $\mathcal{M}^{\geq v}$ over w, i.e., $w \in \left\|\mathcal{M}^{\geq v}\right\|$ as desired. Now, if the second case is true there exist a successful path $P_w = (q_i, a_i, q_{i+1})_{i \geq 0}$ of $\mathcal{N}$ over w such that $wt(q_i, a_i, q_{i+1}) \neq \infty$ for at least one $i \geq 0$. Assume that there exist infinitely many $i \geq 0$ such that $wt(q_i, a_i, q_{i+1}) \neq \infty$. Then, there are also infinitely many $i \geq 0$ with $wt(q_i, a_i, q_{i+1}) = v'' \geq v$, and thus we can define a successful path P''_w of $\mathcal{M}^{\geq v}$ over w of the form $((q_i, j_i, D_i), a_i, (q_{i+1}, j_{i+1}, D_{i+1}))_{i \geq 0}$ with $(q_0, j_0, D_0) = (q_0, 0, C)$, and for all $i \geq 1$, $j_i = 1$, if $j_{i-1} = 0, 1, 3, 4$, and $wt(q_{i-1}, a_{i-1}, q_i) < v$, or $j_i = 3$ if $j_{i-1} = 0, 1, 3, 4$, and $wt(q_{i-1}, a_{i-1}, q_i) \geq v$, or $j_i = 4$ if $j_{i-1} = 0, 1, 3, 4$, and $wt(q_{i-1}, a_{i-1}, q_i) = \infty$, and $D_i = C$ if $q_i \notin F$, and $D_i = B$ if $q_i \in F$. Next, we assume that there exist only a finite number of $i \geq 0$ with $wt(q_i, a_i, q_{i+1}) \neq \infty$. Then, there exist an $h \geq 0$ with

$wt(q_h, a_h, q_{h+1}) = v'' \geq v$, and a $j > h$ with $wt(q_k, a_k, q_{k+1}) = \infty$ for all $k \geq j$. Then, we can define a successful path P''_w of $\mathcal{M}^{\geq v}$ over w as follows

$$\begin{array}{l} P''_w = \left(\left(q_i, j_i, D_i\right), a_i, \left(q_{i+1}, j_{i+1}, D_{i+1}\right)\right)_{0\leq i\leq h-1} \\ \left(\left(q_h, j_h, D_h\right), a_h, \left(s_{q_{h+1}}, 5, D_{h+1}\right)\right) \\ \left(\left(s_{q_i}, 5, D_i\right), a_i, \left(s_{q_{i+1}}, 5, D_{i+1}\right)\right)_{h+1\leq i\leq j-1} \\ \left(\left(s_{q_j}, 5, D_j\right), a_j, \left(s_{q_{j+1}}, 6, D_{j+1}\right)\right) \\ \left(\left(s_{q_k}, 6, D_k\right), a_k, \left(s_{q_{k+1}}, 6, D_{k+1}\right)\right)_{k\geq j+1}, \end{array}$$

where $(q_0, j_0, D_0) = (q_0, 0, C)$. Moreover, for all $0 \leq i \leq h$, $j_i = 1$, if $j_{i-1} = 0, 1, 3, 4$, and $wt(q_{i-1}, a_{i-1}, q_i) < v$, or $j_i = 3$ if $j_{i-1} = 0, 1, 3, 4$, and $wt(q_{i-1}, a_{i-1}, q_i) \geq v$, or $j_i = 4$ if $j_{i-1} = 0, 1, 3, 4$, and $wt(q_{i-1}, a_{i-1}, q_i) = \infty$. Finally, $D_l = C$ if $q_l \notin F$, and $D_l = B$ if $q_l \in F$ for every $l \geq 1$. (We shall call such a path P''_w, a path of form B). Hence, $w \in \left\|\mathcal{M}^{\geq v}\right\|$.

We proved that $\{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\} \subseteq \left\|\mathcal{M}^{\geq v}\right\|$. In order to prove the converse inclusion, we let $P''_w = \left(\left(q_i, j_i, D_i\right), a_i, \left(q_{i+1}, j_{i+1}, D_{i+1}\right)\right)_{i\geq 0}$ with $(q_0, j_0, D_0) = (q_0, 0, C)$ be a successful path of $\mathcal{M}^{\geq v}$ over w. This implies that $In^Q(P''_w) = \widehat{F} \in F^{\geq v}$, i.e., $\widehat{F} \cap F_1 \neq \emptyset$, and $\widehat{F} \cap F_2 \neq \emptyset$. Moreover, by definition of $\Delta^{\geq v}$ we have that $F_1 \cap \widehat{F} \subseteq Q \times \{3\} \times \{C, B\}$ (case 1), or $F_1 \cap \widehat{F} \subseteq \widehat{Q} \times \{2\} \times \{C, B\}$ (case 2), or $F_1 \cap \widehat{F} \subseteq \widehat{Q} \times \{6\} \times \{C, B\}$ (case 3), and $F_2 \cap \widehat{F} \subseteq Q \times \{0, 1, 3, 4\} \times \{B\}$, or $\widehat{F} \cap F_2 \subseteq \widehat{Q} \times \{2, 5, 6\} \times \{B\}$.

Moreover,

$$\left(\left(\widehat{Q} \times \{6\} \times \{C, B\}\right) \cap \widehat{F} \neq \emptyset, \text{ or } \left(\widehat{Q} \times \{2\} \times \{C, B\}\right) \cap \widehat{F} \neq \emptyset\right),$$

implies

$$\widehat{F} \cap \left(\widehat{Q} \times \{2, 5, 6\} \times \{B\}\right) \neq \emptyset,$$

and vice-versa. We assume first that for P''_w case (2), or case (3) holds, then P''_w is either of form (A), or of form (B). If P''_w is of form (A), then $P_w = (q_i, a_i, q_{i+1})_{i\geq 0} \in succ_{\mathcal{N}}(w)$ with $weight_{\mathcal{N}}(P_w) = \infty \geq v$. If P''_w is of form (B), then $P_w = (q_i, a_i, q_{i+1})_{i\geq 0} \in succ_{\mathcal{N}}(w)$ with $weight_{\mathcal{N}}(P_w) = v' \geq v, v' \in \mathbb{Q}$. If case (1) holds, then P''_w is of form (A), and $P_w = (q_i, a_i, q_{i+1})_{i\geq 0}$ is a successful path of $\mathcal{N}$ over w with $weight_{\mathcal{N}}(P_w) = v' \geq v, v' \in \mathbb{Q}$. Thus, $\left\|\mathcal{M}^{\geq v}\right\| \subseteq \{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\}$.

□

Proof: [Proof of Lemma 48] First, we assume that $v \in Q$. We let $\mathcal{M}^{\geq v} = \left(Q^{\geq v}, A, q_0^{\geq v}\Delta^{\geq v}, \mathcal{R}^{\geq v}\right)$ where

- $Q^{\geq v} = (Q \times \{0,1,3,4\} \times \{B,C\}) \cup \left(\widehat{Q} \times \{2,5,6\} \times \{B,C\}\right)$

  where $\widehat{Q} = \{s_q \mid q \in Q\}$ is a set of copies of the elements of Q
- $q_0^{\geq v} = (q_0, 0, C)$
- For all $((q,i,k), a, (\overline{q}, j, l)) \in Q^{\geq v} \times A \times Q^{\geq v}$, it holds $((q,i,k), a, (\overline{q}, j, l)) \in \Delta^{\geq v}$ iff

  1. $i = 0$ and $j = 1$, $k = l = C$, and $q = q_0, \overline{q} \notin F$ and $wt(q, a, \overline{q}) < v$ with $wt(q, a, \overline{q}) \neq -\infty$, or
  2. $i = 0$ and $j = 1$, $k = C, l = B$ and $q = q_0, \overline{q} \in F$ and $wt(q, a, \overline{q}) < v$ with $wt(q, a, \overline{q}) \neq -\infty$, or
  3. $i = 0$ and $j = 3, k = l = C$, and $q = q_0, \overline{q} \notin F, wt(q, a, \overline{q}) \geq v$, and $wt(q, a, \overline{q}) \neq \infty$, or
  4. $i = 0$ and $j = 3, k = C, l = B$, and $q = q_0, \overline{q} \in F, wt(q, a, \overline{q}) \geq v$, and $wt(q, a, \overline{q}) \neq \infty$, or
  5. $i = 0$ and $j = 4, k = l = C$, and $q = q_0, \overline{q} \notin F, wt(q, a, \overline{q}) = \infty$, or
  6. $i = 0$ and $j = 4, k = C, l = B$, and $q = q_0, \overline{q} \in F, wt(q, a, \overline{q}) = \infty$, or
  7. $i = 0$ and $j = 2$, $k = l = C$, and $q = q_0, \overline{q} = s_{\widetilde{q}} \in \widehat{Q}, \widetilde{q} \notin F, wt(q_0, a, \widetilde{q}) = \infty$, or
  8. $i = 0$ and $j = 2$, $k = C, l = B$, and $q = q_0, \overline{q} = s_{\widetilde{q}} \in \widehat{Q}, \widetilde{q} \in F, wt(q_0, a, \widetilde{q}) = \infty$, or
  9. $i = 2$ and $j = 2$, $k \in \{C, B\}, l = B$, and $q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \in F$, $wt(q'', a, \widetilde{q}) = \infty$, or
  10. $i = 2$ and $j = 2$, $k \in \{C, B\}, l = C$, and $q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \notin F$, $wt(q'', a, \widetilde{q}) = \infty$, or
  11. $i = 1, 3, 4$ and $j = 3$, $k \in \{C, B\}, l = B, q \in Q, \overline{q} \in F$, and $wt(q, a, \overline{q}) \geq v$, and $wt(q, a, \overline{q}) \neq \infty$, or
  12. $i = 1, 3, 4$ and $j = 3$, $k \in \{C, B\}, l = C, q \in Q, \overline{q} \in Q \backslash F$, and $wt(q, a, \overline{q}) \geq v$, and $wt(q, a, \overline{q}) \neq \infty$, or
  13. $i = 1, 3, 4$ and $j = 1$, $k \in \{C, B\}, l = B, q \in Q, \overline{q} \in F$, and $wt(q, a, \overline{q}) < v, wt(q, a, \overline{q}) \neq -\infty$, or
  14. $i = 1, 3, 4$ and $j = 1$, $k \in \{C, B\}, l = C, q \in Q, \overline{q} \in Q \backslash F$, and $wt(q, a, \overline{q}) < v, wt(q, a, \overline{q}) \neq -\infty$, or

15. $i = 1,3,4$ and $j = 4$, $k \in \{C,B\}, l = C, q \in Q, \overline{q} \in Q \backslash F$, and $wt(q,a,\overline{q}) = \infty$, or
16. $i = 1,3,4$ and $j = 4$, $k \in \{C,B\}, l = B, q \in Q, \overline{q} \in F$, and $wt(q,a,\overline{q}) = \infty$, or
17. $i = 0$, and $j = 5$, $k \in \{C,B\}, l = C$, and $q = q_0, \overline{q} = s_{\widetilde{q}} \in \widehat{Q}, \widetilde{q} \notin F$, $wt(q_0,a,\widetilde{q}) \geq v$, or
18. $i = 0$, and $j = 5$, $k \in \{C,B\}, l = B$, and $q = q_0, \overline{q} = s_{\widetilde{q}} \in \widehat{Q}, \widetilde{q} \in F$, $wt(q_0,a,\widetilde{q}) \geq v$, or
19. $i = 5$, and $j = 5$, $k \in \{C,B\}, l = C$, and $q, \overline{q} \in \widehat{Q}, q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \notin F, wt(q'',a,\widetilde{q}) \geq v$, or
20. $i = 5$, and $j = 5$, $k \in \{C,B\}, l = B$, and $q, \overline{q} \in \widehat{Q}, q = s_{q''}, \overline{q} = s_{\widetilde{q}}, \widetilde{q} \in F, wt(q'',a,\widetilde{q}) \geq v$.

- The family $\mathcal{R}^{\geq v}$ is defined as follows

$\mathcal{R}^{\geq v} = \left\{ \left(\widehat{S}, \widehat{F}\right) \mid \widehat{F} \in F^{\geq v}, \widehat{S} \subseteq Q \times \{1\} \times \{B,C\} \right\} \cup \left\{ \left(\emptyset, \overline{F}\right) \mid \overline{F} \in \overline{F}^{\geq v} \right\}$
$\cup \left\{ \left(\emptyset, \widetilde{F}\right) \mid \widetilde{F} \in \widetilde{F}^{\geq v} \right\}$ where
$\widehat{F}^{\geq v} = \left\{ \widehat{F} \mid \widehat{F} \subseteq F_1 \cup F_2, F_1 \cap \widehat{F} \neq \emptyset, F_2 \cap \widehat{F} \neq \emptyset \right\}$ with
$F_1 = Q \times \{3\} \times \{C,B\}, F_2 = Q \times \{0,1,3,4\} \times \{B\}$, and
$\overline{F}^{\geq v} = \left\{ \overline{F} \mid \overline{F} \subseteq \overline{F}_1 \cup \overline{F}_2, \overline{F}_1 \cap \overline{F} \neq \emptyset, \overline{F}_2 \cap \overline{F} \neq \emptyset \right\}$ with
$\overline{F}_1 = \widehat{Q} \times \{2\} \times \{C,B\}, \overline{F}_2 = \widehat{Q} \times \{2\} \times \{B\}$, and
$\widetilde{F}^{\geq v} = \left\{ \widetilde{F} \mid \widetilde{F} \subseteq \widehat{Q} \times \{5\} \times \{B\} \right\}$.

We prove now that $\left\| \mathcal{M}^{\geq v} \right\| = \{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\}$. We let first $w = a_0 a_1 a_2 \ldots \in \{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\}$. This implies that

$$(\|\mathcal{N}\|, w) = \sup_{P_w \in succ_{\mathcal{N}}(w)} (weight_{\mathcal{N}}(P_w)) = \infty$$

or

$$(\|\mathcal{N}\|, w) = \sup_{P_w \in succ_{\mathcal{N}}(w)} (weight_{\mathcal{N}}(P_w)) = v'' \in \mathbb{Q} \text{ with } v'' \geq v.$$

If the first case is true there exist a successful path $P_w = (q_i, a_i, q_{i+1})_{i \geq 0}$ of $\mathcal{N}$ over w such that $wt(q_i, a_i, q_{i+1}) = \infty$ for all $i \geq 0$. We consider the path

$$P''_w = ((q_0,0,C), a_0, (s_{q_1},2,D_1)) \left((s_{q_i},2,D_i), a_i, \left(s_{q_{i+1}},2,D_{i+1}\right) \right)_{i \geq 1}$$

where $D_i = B$ if $q_i \in F$, and $D_i = C$ if $q_i \notin F$ for every $i \geq 1$. (We shall call such a path P''_w a path of form A). Clearly, P''_w is a successful path of $\mathcal{M}^{\geq v}$ over w, i.e., $w \in \left\|\mathcal{M}^{\geq v}\right\|$ as desired. Now, if the second case is true there exist a successful path $P_w = (q_i, a_i, q_{i+1})_{i\geq 0}$ of $\mathcal{N}$ over w such that $wt\,(q_i, a_i, q_{i+1}) \neq \infty$ for at least one $i \geq 0$. Assume that there exist infinitely many $i \geq 0$ such that $wt\,(q_i, a_i, q_{i+1}) \neq \infty$. Then, there are also infinitely many $i \geq 0$ with $wt\,(q_i, a_i, q_{i+1}) = v'' \geq v$, and only a finite number of $i \geq 0$ with $wt\,(q_i, a_i, q_{i+1}) < v$. Thus, we can define a successful path P''_w of $\mathcal{M}^{\geq v}$ over w of the form $((q_i, j_i, D_i), a_i, (q_{i+1}, j_{i+1}, D_{i+1}))_{i\geq 0}$ with $(q_0, j_0, D_0) = (q_0, 0, C)$, and for all $i \geq 1$, $j_i = 1$, if $j_{i-1} = 0, 1, 3, 4$, and $wt\,(q_{i-1}, a_{i-1}, q_i) < v$, or $j_i = 3$ if $j_{i-1} = 0, 1, 3, 4$, and $wt\,(q_{i-1}, a_{i-1}, q_i) \geq v$, or $j_i = 4$ if $j_{i-1} = 0, 1, 3, 4$, and $wt\,(q_{i-1}, a_{i-1}, q_i) = \infty$, and $D_i = C$ if $q_i \notin F$, and $D_i = B$ if $q_i \in F$. Next, we assume that there exist only a finite number of $i \geq 0$ with $wt\,(q_i, a_i, q_{i+1}) \neq \infty$. Then, for all $h \geq 0$ with $wt\,(q_h, a_h, q_{h+1}) \neq \infty$, it holds $wt\,(q_h, a_h, q_{h+1}) = v'' \geq v$. Then, we can define a successful path P''_w of $\mathcal{M}^{\geq v}$ over w as follows

$$P''_w = ((q_0, 0, C), a_0, (s_{q_1}, 5, D_1))\,((q_i, 5, D_i), a_i, (q_{i+1}, 5, D_{i+1}))_{i\geq 1}$$

where $D_i = C$ if $q_i \notin F$, and $D_i = B$ if $q_i \in F$ for all $i \geq 1$ (We shall call such a path P''_w, a path of form B). Hence, $w \in \left\|\mathcal{M}^{\geq v}\right\|$.

We proved that $\{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\} \subseteq \left\|\mathcal{M}^{\geq v}\right\|$. In order to prove the converse inclusion, we let $P''_w = ((q_i, j_i, D_i), a_i, (q_{i+1}, j_{i+1}, D_{i+1}))_{i\geq 0}$ with $(q_0, j_0, D_0) = (q_0, 0, C)$ be a successful path of $\mathcal{M}^{\geq v}$ over w. This implies that $In^Q\,(P''_w) \cap \widehat{F} \neq \emptyset$, and $In^Q\,(P''_w) \cap \widehat{S} = \emptyset$ (case 1), or $In^Q\,(P''_w) \cap \overline{F} \neq \emptyset$ (case 2), or $In^Q\,(P''_w) \cap \widetilde{F} \neq \emptyset$ (case 3).

We assume first that for P''_w case (2), or case (3) holds, then P''_w is either of form (A), or of form (B). If P''_w is of form (A), and $P_w = (q_i, a_i, q_{i+1})_{i\geq 0} \in succ_{\mathcal{N}}\,(P_w)$ with $weight_{\mathcal{N}}\,(P_w) = \infty \geq v$. If P''_w is of form (B), then $P_w = (q_i, a_i, q_{i+1})_{i\geq 0} \in succ_{\mathcal{N}}\,(P_w)$ with $weight_{\mathcal{N}}\,(P_w) \geq v$. If case (1) holds, then P''_w is also a successful path of $\mathcal{N}$ over w with $weight_{\mathcal{N}}\,(P_w) = v' \geq v, v' \in \mathbb{Q}$.

Thus, $\left\|\mathcal{M}^{\geq v}\right\| \subseteq \{w \in A^\omega \mid (\|\mathcal{N}\|, w) \geq v\}$, and this concludes our proof.

□